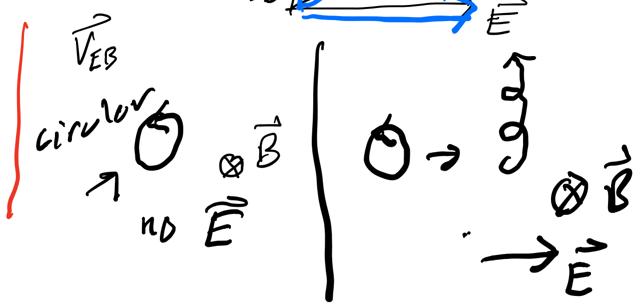
Motion of charged particles in electric and magnetic fields

$$\vec f = q\left(\vec E + \frac{\vec v\times\vec B}{c}\right) = m\frac{d\vec v}{dt}$$

if $\vec v = \vec v\,' + \frac{\vec E\times\vec B}{B^2}$ ⇒ $m\frac{d\vec v\,'}{dt} = q\vec E + q\frac{(\vec E\times\vec B)\times\vec B}{B^2} + q\frac{\vec v\,'}{c}\times\vec B$

$$= \cancel{q\vec E} + \cancel{\frac{q(\vec E\cdot\vec B)\vec B}{B^2}} - \cancel{\frac{q\vec E(\vec B\cdot\vec B)}{B^2}} + q\frac{\vec v\,'\times\vec B}{c}$$

⇒ $\vec V_{EB} = \frac{\vec E\times\vec B}{B^2}$ ⇒ $\vec E\times\vec B$ drift



• In frame of $\vec V_{EB} = 0$:

$$\frac{d\vec v\,'}{dt} = \frac{q\vec v\,'\times\vec B}{mc}$$

or $\boxed{\frac{d\vec v}{dt} = q\frac{\vec v\times\vec B}{mc}}$ for $\vec E = 0$;

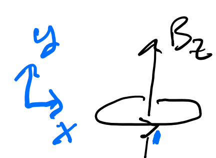
assume $\vec B = B_z\hat z$

$\omega = \left|\frac{qB_z}{mc}\right|$

(*) $\frac{d\;\;x}{dt} = \mp \quad y$

$\frac{dV_y}{dt} = \pm\,\omega V_x$ $\begin{pmatrix} q<0 \\ q>0 \end{pmatrix}$

⇒ $$\boxed{\frac{d^2V_x}{dt^2} = -\omega^2 V_x \qquad \frac{d^2V}{dt^2} = -\omega^2 V_y}$$



(electron example)

$V_x(0) = U_0$

$V_y(0) = 0$

⇒ $V_x = a_1\cos\omega t + b_1\sin\omega t = U_0\cos\omega t + \cancel{b_1}^{\,0}\sin\omega t$

electron $V_y = a_2\cos\omega t + b_2\sin\omega t = U_0\sin\ \ t$; $b_2 = 0$

b dry cond. $\{\dot V_y(0) = -\ \ V_x(0)$; $\dot V_x(0) = \omega V_y(0)$; $= 0$; $V_x(0) = U_0\}$

$\Rightarrow \quad b_1 = 0$, since $\frac{dV_x}{dt} = -u_0 \sin\omega t + \underbrace{b_1}_{\text{0 to match b.c. at } t=0} \omega\cos\omega t$

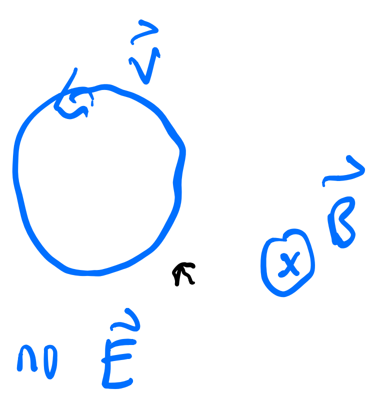
$$\Rightarrow \boxed{\begin{aligned} V_x &= u_0 \cos\omega t \\ V_y &= u_0 \sin\omega t \end{aligned} \quad \Rightarrow \quad \begin{aligned} x &= \frac{u_0}{\omega}\sin\omega t \\ y &= -u_0/\omega \cos\omega t \end{aligned}} \qquad (209\ \text{a--d})$$

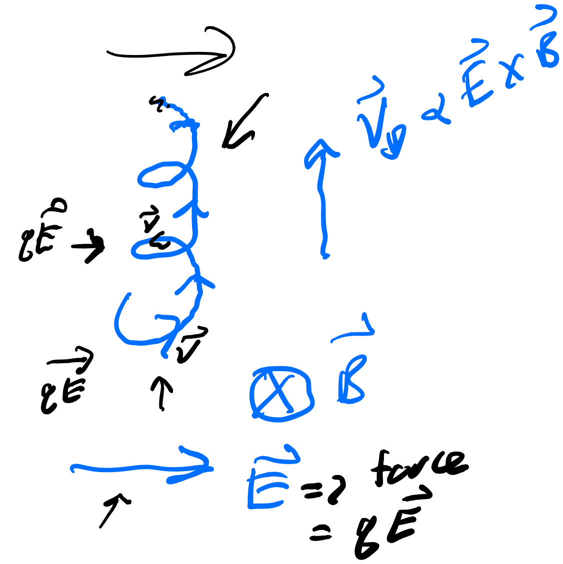
- Drift will occur not only for $\vec{E}$, but for any $\vec{F} \perp \vec{B}$

$$\Rightarrow \boxed{\vec{V}_{drift} = \frac{\vec{F}/q \times \vec{B}}{\vec{B}^2}} \qquad (215)$$

($\vec{F}_{\nabla B} \propto \nabla B$ is another example)

lets see $\nabla\vec{B}$ drift next:

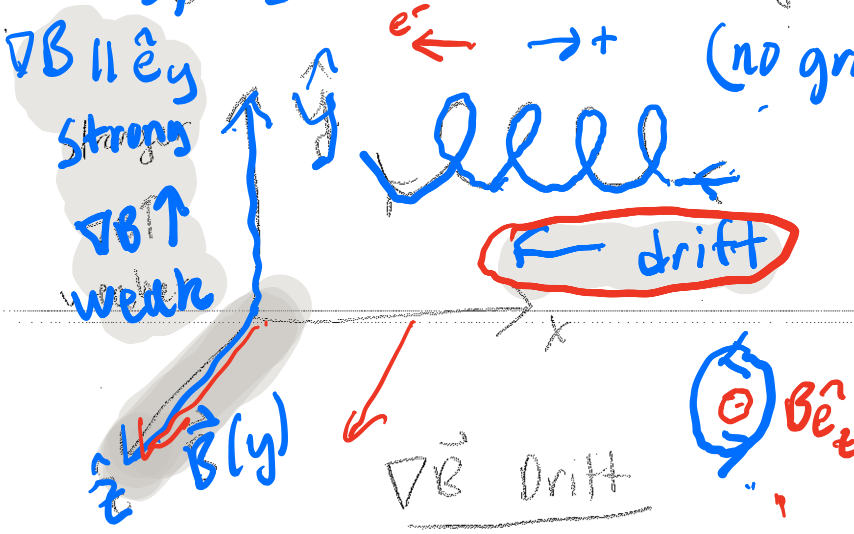




∇B Drift:

Consider uni-directional magnetic field $B(y)\,\hat{e}_z$ where strength depends on y

(no gradient $\Rightarrow$ ○)



the particle has tighter wind where the field is stronger (at the top) thus the particle ... drifts. $\boxed{\text{This is } \nabla B \text{ Drift}}$

Lets calculate the time averaged force in the y direction. The y-force is

$F_y = -\frac{q}{c} u_x B_z(y)$, where y is measured from position of the guiding center.

For variation $B_z(y)$ small over the distance of particle trajectory in y-direction, to lowest order

$$B_z(y) = B_0 + y\frac{dB_z}{dy} + \ldots \qquad (217)$$

thus
$$\langle F_y\rangle = -\left\langle \frac{q}{c} u_x \left[B_0 + y\frac{dB_z}{dy}\right]\right\rangle_{\text{orb.}} \qquad (218)$$

average this

Now let us average over an orbit and assume that over this time scale the particle moves in nearly circular orbit $\rightarrow$

The average of the first term in (218) vanishes since $\langle u_x B_0 \rangle = \langle u_x \rangle B_0 = 0$ over a cycle. Averaging the second term in (204) gives

pulled out assuming $|r_g||\nabla| < 1$

$$\langle F_y \rangle \underset{GS}{=} \left\langle -\frac{q}{c} u_x y \right\rangle \frac{dB_z}{dy} \tag{219}$$

$$|\omega_e| = \left|\frac{q B_0}{mc}\right| = \left|\frac{e B_0}{m_e c}\right|$$

but over a given cycle,

$$-\left\langle \frac{q}{c} u_x y \right\rangle = +\frac{q}{c} \int_0^{2\pi/\omega} dt \, \frac{u_0^2}{\omega} \cos^2 \omega t \Big/ \int_0^{2\pi/\omega} dt \tag{220}$$

since from (209 a–d) we have used

$u_x = \pm u_0 \cos \omega t$, and $y = \mp \frac{u_0}{|\omega|} \cos \omega t$

Thus (220) $\Rightarrow$

$$\langle F_y \rangle = \left\langle -\frac{q}{c} u_x y \right\rangle \frac{dB_y}{dy} = \mp \frac{1}{2} \frac{q}{c} \frac{u_0^2}{|\omega|} \nabla_y B_z(y) \tag{221}$$

where $\dfrac{\int_0^{2\pi/\omega} \cos^2 \omega t \, dt}{\int_0^{2\pi/\omega} dt} = \frac{1}{2}$ was used, and

$\mp$ depends on sign of charge. ($-$ for $+$ charge, $+$ for $-$ charge)

Using (221) in (215) gives

$$\vec{u}_{\nabla B, gc} = \pm \frac{1}{2}\frac{u_0^2}{|\omega|}\frac{\vec{B}\times\vec{\nabla} B}{B^2}$$

$$\frac{u_0}{\omega} = r_g \omega$$

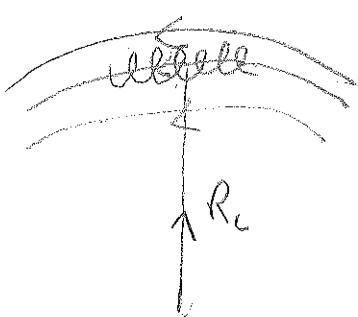
here + for + charge, − for − charge

$$\vec{u}_{\nabla B} = \boxed{\pm \frac{1}{2} u_0 r_g \frac{\vec{B}\times\vec{\nabla}\vec{B}}{B^2}} \qquad (222)$$

where u_0 is circular speed and r_g is gyrofrequency, $r_g = \frac{mcu_0}{|Ze|B_z}$ charge $q \equiv Ze$

Eq'n (222) gives $\vec{\nabla}B$ Drift velocity. Note opposite signs for opposite charges $\Rightarrow$ even a current when both signed charges are present.

Curvature Drift: Consider nearly constant, but curved magnetic field:



effective central force

$$\vec{F}_c = -\frac{m u_\parallel^2}{|R_c|}\hat{R}_c \qquad (223)$$

$u_\parallel$ is velocity $\parallel$ to B, R_c is radius of curvature. Again $\vec{F}$ is $\perp$ to $\vec{B}$ so from (215)

$$\vec{u}_{c,gc} = -\frac{c m u_\parallel^2}{qB^2}\frac{\hat{R}_c\times\vec{B}}{|R_c|} \qquad (224)$$

Shocks: $\boxed{\text{Does } \vec{E}\times\vec{B} \text{ or } \vec{\nabla}B\times\vec{B} \text{ drift dominate?}}$

$$\frac{q}{c}\frac{u_0^2}{w}\nabla\vec{B} = \vec{F}_{\vec{\nabla}B}$$

particle velocity

$$= \frac{q\, u_0\, r_g}{c}\nabla \vec{B}$$

$$\frac{F_{\vec{\nabla}B}}{F_{\vec{E}}} = \frac{F_{\vec{\nabla}B}}{q\vec{E}}$$

ratio

at shock $|\nabla| \simeq \frac{1}{r_{g,i}}$. So for ions,

$$\boxed{|F_{\vec{\nabla}B}| \simeq \left|\frac{q\, u_{0,i}}{c}B\right| \cdot \frac{1}{3}}$$ flow velocity

But $\boxed{q\vec{E} = -q\frac{\vec{v}_f}{c}\times\vec{B}}$, where $\vec{v}_f$ is flow velocity

but $|\vec{v}_f| << |u_0|$ for high enough energy particles, so $\boxed{\nabla B \text{ drift}}$ $\boxed{\text{dominate ions}}$. For electrons:

$$|F_{\vec{\nabla}B}| = \left|q\, u_{0,e}\frac{r_{g,e}}{r_{g,i}}B\right|$$ so then

$$r_{g,e} = \frac{m_e c u_{0,e}}{eB} \qquad r_{g,i} = \frac{m_i c u_{0,i}}{eB} \Rightarrow \boxed{\frac{r_{g,e}}{r_{g_i}} = \frac{\cancel{m_e u_{0,e}^2}}{\cancel{m_i u_{0,i}^2}} \frac{u_{i,0}}{u_{e,0}}}$$

BUT if initial thermal equilib:

then $m_e u_{0,e}^2 \simeq m_i u_{0,i}^2$

$$\frac{F_{\nabla B,i}}{F_{E,i}} \simeq \frac{u_{0,i}}{v_f}$$

but for electrons

$$\boxed{\frac{F_{\nabla B,e}}{F_{E,e}} = \frac{u_{0,e} \cancel{\frac{r_{g,e}}{r_{g,i}}}}{v_f}} = \boxed{\frac{u_{0,i}}{v_f}}$$

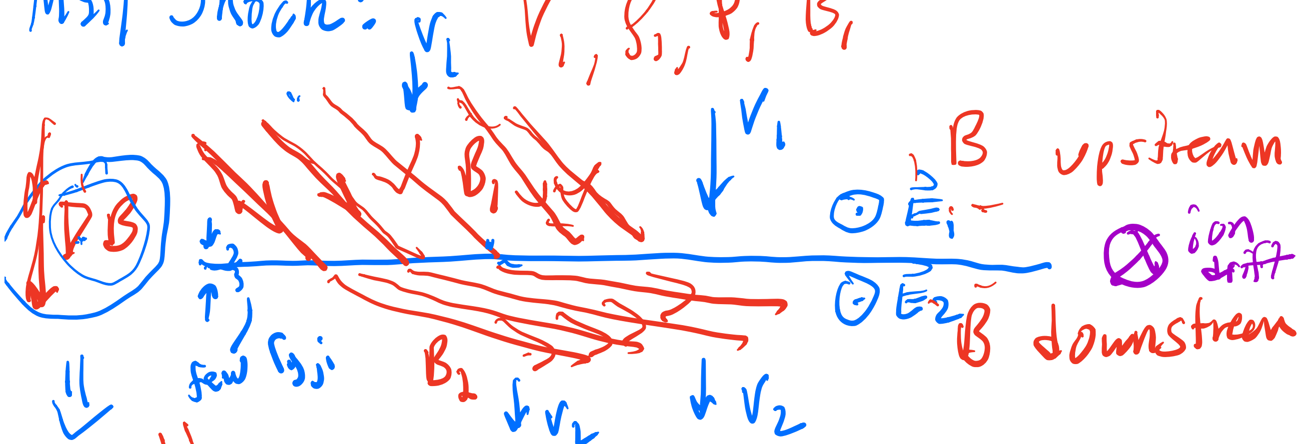
$= u_{0,i}$

$$F_{drift,\nabla B,e} \approx \frac{e\, u_{o,e}}{c} \, \frac{r_{g,e}\, B}{r_{g,i}}$$

$$\sim \frac{e}{c}\, u_{o,e} \frac{r_{g,e}}{r_{g,i}} \, B$$

compare to V_f

MHD Shock: V_1, ρ_1, P_1, B_1



$\Downarrow$ "fast mode shock" $B_{down} > B_{up}$

$|\nabla B| \sim \left|\frac{1}{(\text{few } r_{g,i})}\right|$

V_2, ρ_2, P_2, B_2 ⇒ shock has large $\underline{\nabla B}$

$\rho_2 = 4\rho_1$

Also: $\vec{E}_1 = -\vec{V}_1 \times \vec{B}_1 = \vec{E}_2 = -\vec{V}_2 \times \vec{B}_2$

Maxwell's B.C. ⇒

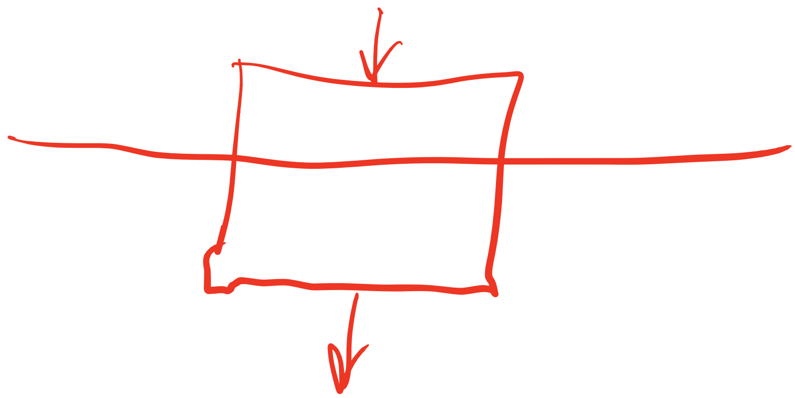
$$\partial_t \vec{V} = -\vec{v} \cdot \nabla \vec{v} - \frac{\vec{\nabla} P}{\rho} + \boxed{\frac{\vec{J} \times \vec{B}}{\rho}} + \underbrace{\rho_{charge} \vec{E}}_{0}$$

$\rho_2 \simeq 4\rho_1 \Rightarrow V_2 = \frac{1}{4} V_1$

$\rho_1 V_1 = \rho_2 V_2$

$B_{1n} \approx B_{2n}$

$B_{1t} < B_{2t}$ from compression



$$\left.\begin{array}{ll} \text{Both} & \vec{E}_1 \times \vec{B}_1 \neq 0 \\ \text{and} & \nabla B_1 \times \vec{B}_1 \neq 0 \end{array}\right\} \Rightarrow \underline{\text{drifts}}$$

ratio of drift forces is:

$$\frac{|qE|}{\left|q u_{0,e} \frac{r_{g,e}}{r_{g,i}}\right|} = \frac{V_f}{u_{0,e}} \frac{r_{g,i}}{r_{g,e}} = \frac{V_f}{u_{0,e}} \frac{m_i}{m_e} \frac{u_{0,i}}{u_{0,e}} \quad (*)$$

for plasma initially in thermal equilibrium:

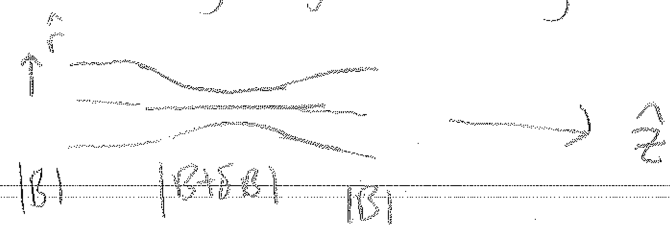
$$\frac{m_i u_{0,i}^2}{m_e u_{0,e}^2} = 1, \quad \text{so } * \Rightarrow$$

$$\frac{V_f}{u_{i,0}} \Rightarrow \text{again } \nabla \vec{B} \text{ drift}$$

<u>dominates</u>

Magnetic Mirroring

Consider magnetic field such that the strength varies axially symmetrically along the field line:



Assume field increase is such that $\delta B/B < 1$.

$\nabla \cdot B = 0$ in cylindrical coordinates gives

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0 \qquad (225)$$

Assume $\frac{\partial B_z}{\partial z} \simeq$ constant near $B = B_{max}$;

then we can integrate (225)

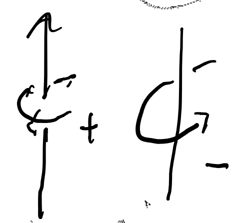
$$r B_r = -\int_0^r r' \frac{\partial B_z}{\partial z}\, dr' \simeq -\frac{\partial B_z}{\partial z}\int_0^r r' dr' = -\frac{1}{2} r^2 \frac{\partial B_z}{\partial z}$$

$$\Rightarrow B_r = -\frac{1}{2} r \frac{\partial B_z}{\partial z} \qquad (226)$$

For particle gyrating on field line, z-component of Lorentz force is

$$F_z = -\frac{q}{c} u_\theta B_r \qquad (227)$$

Positive (negative) 'charges' move around the field lines in the $-\theta$ ($+\theta$) direction



clockwise (counter clockwise)

so $u_\theta = \mp u_\perp$. Then using 226 in 227

gyro-radius

$$F_z = \mp \frac{q}{2c} u_\perp r_g \frac{\partial B_z}{\partial z} \equiv -\mu \frac{\partial B_z}{\partial z} \simeq -\mu \frac{\partial B}{\partial z}$$

$$\mu = \pm \frac{q}{2c} u_\perp \overbrace{\left(\frac{u_\perp m c}{qB}\right)}^{r_g} = \pm \frac{1}{2} \frac{m u_\perp^2}{B} \qquad (228)$$

(gyroradius r_g enters for r in (226) because only B_r within gyroradius of orbiting particle matters)

We can also eliminate $u_\perp$ instead of r_g using $r_g = \frac{u_\perp mc}{qB} = \frac{u_\perp}{\omega_c}$:

$$\mu = \pm \frac{q r_g^2 \omega_c}{2c} = \pm \underbrace{\frac{\omega_c}{2\pi} q}_{\text{current associated with gyration}} \underbrace{\left(\pi r_g^2\right)}_{\text{area around which current encircles}} \frac{1}{c} = \frac{IA}{c}$$

units: $\left[\frac{IA}{c}\right] = \left[\frac{B\cdot A^2}{\ell}\right] \Rightarrow \mu\cdot B$ = energy

μ is <u>magnetic moment</u>.

The z-component of motion is

$$m \frac{du_\parallel}{dt} = F_z = -\mu \frac{\partial B}{\partial z} \qquad (229)$$

the rate of change in kinetic energy associated with velocity along the field line is

$$\frac{m}{2}\frac{d u_{\parallel}^2}{dt} = u_{\parallel} m \frac{d u_{\parallel}}{dt} = \mu_1 \frac{dB}{dt} \qquad (230)$$

$$\left(\text{since} \quad \frac{dB}{dt} = \underbrace{\cancel{\frac{\partial B}{\partial t}}}_{0} + \underbrace{\vec{u}\cdot\vec{\nabla} B}_{= u_{\parallel}\partial_z B + u_\theta \underbrace{\cancel{\partial_\theta} B}_{0} + \underbrace{u_r \cancel{\partial_r} B}_{\simeq 0}} \right)$$

Since total particle energy cannot change in steady magnetic field $\left(\frac{du^2}{dt} = u\cdot(u\times\vec{B}) = 0\right)$

$$\frac{d}{dt}\left(\frac{1}{2}mu_{\parallel}^2\right) + \frac{d}{dt}\left(\frac{1}{2}m u_{\perp}^2\right) = 0, \quad \text{from (230) \& (228):}$$

$$\Rightarrow -\mu\frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0$$

$$\Rightarrow \frac{d\mu}{dt} = 0 \qquad (231)$$

$\Rightarrow$ magnetic moment is conserved during the motion of the particle's guiding center $\rightarrow$

this implies $\frac{\frac{1}{2}mu_\perp^2}{B} = \mu =$ constant

so $\frac{1}{2}mu_\perp^2$ must increase as B increases.

But $\frac{1}{2}mu_\perp^2$ cannot increase more than the total kinetic energy, thus the particle can no longer proceed into regions of stronger B at that point, and its only option is to reflect back or "mirror". Thus the field can act as a magnetic mirror.

Note however that if particles have small perpendicular velocity to begin with, they are unaffected by magnetic forces and do not participate in the mirroring. To quantify this, we can compute the maximum angle α_m that a particles velocity can make with the symmetry axis and still penetrate the mirror.

Let $\vec{u}_0$ be velocity of particle at P in the above figure. Then

$$u_{\perp 0} = u_0 \sin \alpha_m \quad (232)$$

If B is the mag field at P, then μ being constant for particles that reflect implies

$$\frac{u_{\perp 0}^2}{B} \geq \frac{u_0^2}{B_m} \quad (233)$$

$\leftarrow$ field at maximum (point Q in figure)

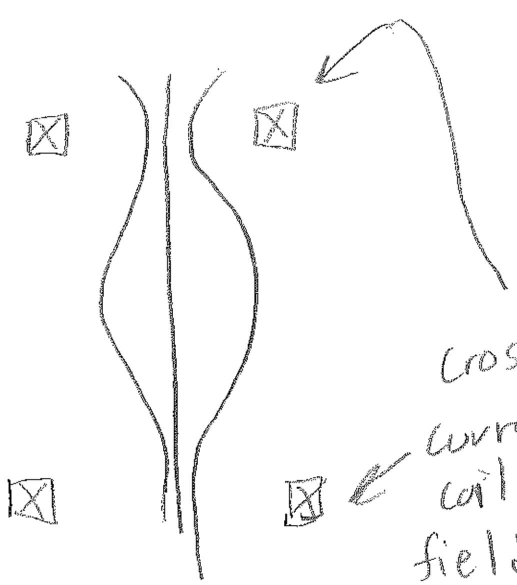
for those particles that reflect at Q. (because $|u| = |u_0|$ at the moment of reflection) Then (232) and (233) give

$$\sin^2 \alpha_m \geq \frac{B}{B_m} \text{ for reflection} \quad (234).$$

Particles with $\sin \alpha_m < \frac{B}{B_m}$, never reflect, since their $\frac{u_{\perp 0}^2}{B} < \frac{u_0^2}{B_M}$, and $\frac{u_0^2}{B_m}$ is the smallest $\frac{\mu}{m}$ the particle can have and still be reflected. The particles with $\sin \alpha < \sin \alpha_m$ at P are not reflected. They pass through the mirror and are "lost" and α_m defines the loss cone. (Particles are "lost" in the sense that they are not trapped by the mirror, passing right through.

Van Allen Belts

Consider application of the concepts of previous sections on orbit theory and mirroring. Consider region with two magnetic mirrors at each end; we can form a magnetic bottle:



cross section of current ring coils to enhance field

- Magnetic Bottle - can be used to trap and confine particles in Laboratory Devices.
- Fermi acceleration - moving magnetic mirrors.

Now consider configuration outside spherical dipole:

Field is pinched at Q_1 & Q_2 and thus this acts as a magnetic bottle too, but that is not all →

Note that because of curved field lines and because of radial field gradient, particles will also incur curvature and ∇B drift forces.

These act in direction $\hat{r} \times \vec{B}$ and $\nabla B \times \vec{B}$ respectively. Both act to give particle motion a component azimuthally around $\hat{z}$, that is, $\perp$ to both $\hat{r}$ & $\hat{z}$. In a general dipole field motion can be somewhat complicated. Nevertheless, this is exactly how the van Allen Belts of trapped charged particles in the earth's magnetosphere arise. The gradient & curvature forces produce an azimuthal current called the ring current. Current is produced because ∇B and curvature drift forces send e^- and protons in opposite directions.