Viscous evolution equations for Accretion Disks

Rather than try to "construct" viscous transport from first principles as attempted (and done very incorrectly in some textbooks) lets <u>assume</u> that turbulence acts as a viscosity to then derive the accretion disk transport equations. Note that this "assumption" is really equivalent to what is currently used in disk modeling for direct comparisons with observations but not a fundamentally consistent or complete approach. It is a theoretical frontier to improve the theory.

So for the present we will explicitly assume a closure for which the Reynolds stess terms associated with turbulent fluctuations Navier stokes equation take the form:

$$\overline{\vec{u}' \cdot \nabla \vec{u}'} = -\nabla \times (\nu_T \nabla \times \bar{u}) \qquad (1r)$$

(closure to obtain standard acc. disk theory as mean field theory)

where $\vec{u} = \vec{u}' + \bar{u}$
fluctuation, mean

$\nu_T = \frac{\bar{\Sigma}}{2H} \eta \simeq v_T l_T$ = turbulent viscosity

kinematic viscosity

See next page for $\bar{\Sigma}$ and H: →

The continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \qquad \text{for } \rho = \rho(r,\phi,z),\ u = u(r,\phi,z) \qquad (2r)$$

define $$\bar{\Sigma} = \underbrace{\frac{\int_{-H}^{H}\int_{0}^{2\pi} \rho \, d\phi \, dz}{2\pi}}_{\text{mean surface density}}$$ and $$\bar{u} = \frac{\int_{-H}^{H}\int_{0}^{2\pi} \rho u \, d\phi \, dz}{\int_{-H}^{H}\int_{0}^{2\pi} \rho \, d\phi \, dz} = \frac{\int_{-H}^{H}\int_{0}^{2\pi} \rho u \, d\phi \, dz}{2\pi \bar{\Sigma}}$$

(H is 1/2 thickness of disk) = density weighted mean velocity

$\Rightarrow \quad \bar{\Sigma} = \bar{\Sigma}(r), \quad \bar{u} = \bar{u}(r)$ (z, ϕ are averaged out)

then after integrating over $d\phi\,dz$, (1q) $\Rightarrow$

$$\frac{\partial \bar{\Sigma}}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R \bar{\Sigma} U_R\right) = 0 \quad \text{(cylind. coords)} \qquad (3r)$$

Similarly, from the ϕ component of Nav.-Stokes:

$$\bar{\Sigma}\left(\frac{\partial \bar{u}_\phi}{\partial t} + \bar{u}_R \frac{\partial \bar{u}_\phi}{\partial R} + \frac{\bar{u}_R \bar{u}_\phi}{R}\right) = \frac{\partial}{\partial R}\left(\nu_T \bar{\Sigma} \frac{\partial \bar{u}_\phi}{\partial R}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\nu_T \bar{\Sigma} \bar{u}_\phi\right) - \frac{\nu_T \bar{\Sigma} \bar{u}_\phi}{R^2} - \frac{2 u_0}{R}\frac{\partial \Omega}{\partial R} \qquad (4r)$$

Hereafter for notational simplicity I drop the overbars on $\bar{u}$, $\bar{\Sigma}$ and write $\nu_T = \nu$. That is $\bar{u} \to u$ and $\bar{\Sigma} \to \Sigma$.

Then multiply eqn 3r by $R u_\phi$:

$\Rightarrow$

$$R u_\phi \frac{\partial \Sigma}{\partial t} + u_\phi \frac{\partial}{\partial R}\left(R \Sigma u_R\right) = 0 \qquad (5r)$$

and multiply Eqn (3r) by R:

$\Rightarrow$

$$R\Sigma \frac{\partial u_\phi}{\partial t} + R \Sigma u_R \frac{\partial u_\phi}{\partial R} + \Sigma u_R u_\phi$$
$$= R \frac{\partial}{\partial R}\left(\nu \Sigma \frac{\partial u_\phi}{\partial R}\right) + \frac{\partial (\nu \Sigma u_\phi)}{\partial R} - \frac{\nu \Sigma u_\phi}{R} - \frac{\partial u_\phi}{} \frac{\partial \eta}{\partial R}$$

(6r)

next page

Footnote: the ϕ component of the axisymmetric Navier stokes equation equation that arises if one assumes $\frac{\partial}{\partial \phi} = 0$ of all quantites and assumes $u = \bar{u}$ and $\rho = \bar{\rho}$ and simply replaces $\rho\eta$ with $\bar{\rho}\eta$ is

$$\bar{\rho}\left(\frac{\partial \bar{u}_\phi}{\partial t} + \bar{u}_R \frac{\partial \bar{u}_\phi}{\partial R} + \frac{\bar{u}_R \bar{u}_\phi}{R}\right) = \frac{\partial}{\partial R}\left(\eta \frac{\partial \bar{u}_\phi}{\partial R}\right) + \frac{\partial}{\partial z}\left(\eta \frac{\partial \bar{u}_\phi}{\partial z}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\eta \bar{u}_\phi\right) - \frac{\bar{u}_\phi \eta}{R^2} - \frac{\partial \bar{u}_\phi}{R} \frac{\partial \eta}{\partial R}$$

6r can be derived by by integrating this over z. Often the distinction between u and $\bar{u}$ is incorrectly ignored so one should really formally average

(42r)

Add (5r) + (6r) using $\Omega = \dfrac{u_\phi}{R}$

$\Rightarrow$ ⊙ Hη

$$\underbrace{R\frac{\partial(\Sigma u_\phi)}{\partial t}}_{\frac{\partial(\Sigma u_\phi R)}{\partial t}} + \overbrace{\frac{\partial}{\partial R}\left(R\Sigma u_R u_\phi\right) + \Sigma u_R u_\phi}^{\frac{1}{R}\frac{\partial}{\partial R}\left(R^2\Sigma u_R u_\phi\right)} = R\frac{\partial}{\partial R}\left(\nu\Sigma\frac{\partial}{\partial R}(\Omega R)\right)$$

$$+ \frac{\partial}{\partial R}(\nu\Sigma\,\Omega R) - \nu\Sigma\Omega$$

$$+ \frac{2\Omega R}{R}\frac{\partial \Sigma\nu}{\partial R}$$

$\therefore$ $$\frac{\partial(\Sigma u_\phi R)}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}\left(R^2\Sigma u_R u_\phi\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma\left(R^3\frac{\partial\Omega}{\partial R} + \Omega R^2\right)\right) \quad \text{A} \quad \text{B} \quad (7r)$$

B + C + D + E = 0

$$- 2\nu\Sigma\cancel{\frac{\partial}{\partial R}}(\Omega R) \quad \text{C}$$

$$+ R\cancel{\frac{\partial}{\partial R}}(\nu\Sigma\Omega) \quad \text{D}$$

$$- 2\Omega R\cancel{\frac{\partial(\Sigma\nu)}{\partial R}} \quad \text{E}$$

$\Downarrow$

$$= \frac{1}{R}\frac{\partial\left(\nu\Sigma R^3\frac{\partial\Omega}{\partial R}\right)}{\partial R} + 0$$

$$\frac{\partial}{\partial t}\left(R\Sigma u_\phi\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(\Sigma R^2 u_\phi u_R\right) = \underbrace{\frac{1}{R}\frac{\partial}{\partial R}\left(\nu\Sigma R^3 \frac{d\Omega}{dR}\right)}_{\text{viscous } \frac{\text{torque}}{\text{area}}} \quad (8r)$$

Eulerian change of ϕ momentum per area

divergence of flux of ϕ momentum per area

$H\eta R^3 \frac{d\Omega}{dR}$

$-\frac{3}{2} H\eta R^3$

$-\frac{3}{2} H\eta R^3 \Omega$

multiply both sides by $2\pi R dR$ so that equation represents angular momentum evolution of an annulus.

$$dR\frac{\partial}{\partial t}\left(2\pi R^2 \Sigma u_R\right) + \frac{dR}{\cancel{R}}\frac{\partial}{\partial R}\left(2\pi R^2 \Sigma u_\phi u_R\right) = \underbrace{dR\frac{\partial}{\partial R}\left(2\pi\nu\Sigma R^3 \frac{\partial\Omega}{\partial R}\right)}_{\text{net viscous torque on annulus.}} \quad (9r)$$

$$dR\frac{\partial G}{\partial R} = dG$$

torque at radius R:

$\Rightarrow$ $G(R) = 2\pi\nu\Sigma R^3 \frac{\partial\Omega}{\partial R}$ ← (Eqn. 10r)

$= R \times (\text{viscous force})$

$= R\left(2\pi\nu\Sigma R^2 \frac{\partial\Omega}{\partial R}\right)$

$= R\left(2\pi\nu\, 2\rho H\, R^2 \frac{\partial\Omega}{\partial R}\right)$ note: disk thickness $= 2H$

$= R\left(4\pi R H \underbrace{\rho\nu R \frac{\partial\Omega}{\partial R}}\right) = R(2\times\text{Area of annulus})(\sigma_{r\phi}) = \text{torque}$

$\sigma_{r\phi}$ = force per unit area in tangential direction acting on surface with radial normal

Check physical consistency:

$G = 0$ for $\frac{d\Omega}{dR} = 0$ ✓

Che

$G < 0$ for $\frac{d\Omega}{dR} < 0$ ✓

total torque on ring of gas between R, $R+dR$:

$G(R+dR) - G(R) = \frac{\partial G}{\partial R} dR = dG$. Now

rate of work $= d\vec{F} \cdot \vec{v} \simeq d\vec{F} \cdot (\vec{\Omega} \times \vec{R})$

$= \vec{\Omega} \cdot (\vec{R} \times d\vec{F})$

$= \vec{\Omega} \cdot d\vec{G} = \pm \Omega\, dG$

$\Rightarrow$ rate of work (because $d\vec{G} \parallel \pm\vec{\Omega}$)

$= \Omega \frac{\partial G}{\partial R} dR = \frac{\partial(\Omega G)}{\partial R} dR - G \frac{\partial \Omega}{\partial R} dR$

integrate: $\Rightarrow$ total work rate

$$= \underbrace{\int_{R_{in}}^{R_{out}} \frac{\partial(\Omega G)}{\partial R} dR}_{\text{boundary term}} - \underbrace{\int_{R_{in}}^{R_{out}} G \frac{\partial \Omega}{\partial R} dR}_{\text{internal dissipation term}} \rightarrow$$

dissipation term converts mechanical energy into particle energy → heat → radiation

per area (2 faces of ring) =>

2 faces →

$$\boxed{\frac{G \frac{\partial \Omega}{\partial R} \cancel{dR}}{4\pi R \cancel{dR}} = \frac{G(R)}{4\pi R}\frac{d\Omega}{dR} = +\frac{1}{2}\nu \Sigma R^2 \left(\frac{d\Omega}{dR}\right)^2 \left(\text{from } \frac{(1ar)}{(\text{page } 143)}\right) = D(R) = \text{energy loss rate per unit area from dissipation}}$$

note we need to have $\frac{d\Omega}{dR} \neq 0$

need to know ν, Σ to compare to observations:

→

Viscosity can be estimated by characteristic velocity and length scale associated with particle motions & deflections.

The <u>force density</u> associated with the viscosity of the previous section comes from the $\rho \nu \nabla^2 V$ term in navier stokes equation. Recall that $\nu = \nu_T + \nu_{microphys}$ with $\nu_T >>> \nu_{microphys}$

to recall its importance we can compute the Reynolds number: ratio of $V \cdot \nabla V$ term to $\nu \nabla^2 V$ term

for $V \simeq V_\phi$, $\nabla \sim \frac{1}{R}$, $\nu \simeq l_T V_T + \cancel{l_{mfp} C_s}_{microphys}$ small

$$\Rightarrow \frac{|V \cdot \nabla V|}{|\nu \nabla^2 V|} \simeq \frac{R V_\phi}{l_T V_T} = R_{e,eff} \simeq 1 \longrightarrow$$

Note: If turbulence were absent, recall that l = microphysical deflection scale from coulomb collisions for protons

$C_s \simeq$ average proton speed

$$R_{e_{micro}} = \frac{R V_\phi}{\nu_{micro}} \simeq 10^{14} \left(\frac{n}{10^{15}}\right) \left(M/M_\odot\right)^{1/2} \left(\frac{R}{10^{10} cm}\right)^{1/2} \left(\frac{T}{10^4 K}\right)^{-5/2} << R_{e,eff}$$

Thus V_T is associated with Macroscopic, instead of microscopic values.

Shakura & Sunyaev (1973) parameterized $\nu_T = \ell V_T = \alpha_{ss} c_s H$

where H is disk height, c_s is sound speed and α_{ss} is parameter.

$\alpha_{ss} < 1$ under assumption that, for disk which is vertically pressure supported, maximum random velocity is c_s, (more on that later). Also, any structure must be < disk height H. Thus $\alpha_{ss} \leq 1$, determining its exact value is an ongoing struggle

leading model is turbulence generated by magneto-rotational instability (e.g. Balbus & Hawley, Rev Mod Phy 1998)

(Note also Blackman et al. 2006 for relation between α and $\beta = \frac{P_{th}}{B^2/8\pi}$: robust in many sims: $\alpha = 0.2/\beta$)