Apr

Basic Magnetohydrodynamics (cont)

The momentum equation as derived last time, now has the additional $\frac{\vec{J}\times\vec{B}}{c}$ term. This magnetic force can be re-written using $\nabla\times B = \frac{4\pi\vec{J}}{c}$ (From maxwell's equations for non-relativistic flows.)

Thus:

$$\vec{J}\times\vec{B} = \frac{\cancel{c}}{4\pi}\frac{j\times B}{\cancel{c}} = \frac{1}{4\pi}(\nabla\times B)\times B \qquad (280)$$

$$= \frac{1}{4\pi}(\varepsilon_{ijk}\partial_j B_k)\,\varepsilon_{min}B_n$$

$$= \frac{1}{4\pi}(\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn})\,B_n\,\partial_j B_k$$

$$= \frac{1}{4\pi}\vec{B}\cdot\nabla\vec{B} - \frac{1}{8\pi}\nabla B^2 \qquad (281)$$

Thus we can write the MHD momentum equation

$$\rho\frac{\partial\vec{v}}{\partial t} = -\vec{v}\cdot\nabla\vec{v} - \nabla(P + \underbrace{P_{mag}}_{=\frac{B^2}{8\pi}}) + \frac{(\vec{B}\cdot\vec{\nabla})\vec{B}}{4\pi} + \nu\nabla^2\vec{v} + \vec{F} \qquad (282)$$

acts as additional pressure; what about $\frac{\vec{B}\cdot\nabla\vec{B}}{4\pi}$? →

We can show that $\vec{B}\cdot\nabla\vec{B}$ acts as a tension force: consider the tensor M_{ij} defined such that

$$M_{ij} = \frac{B^2}{8\pi}\delta_{ij} - \frac{B_i B_j}{4\pi} \qquad (283)$$

so that

$$\frac{(J \times B)_i}{c} = -\partial_j M_{ij} \qquad (284)$$

from (281), and $\vec{\nabla}\cdot\vec{B} = \partial_i B_i = 0$.

Suppose we choose the $\hat{z}$ axis as the local direction of the magnetic field. Then from (283):

$$M_{ij} = \begin{pmatrix} B_z^2/8\pi & 0 & 0 \\ 0 & B_z^2/8\pi & 0 \\ 0 & 0 & -\frac{B_z^2}{4\pi} \end{pmatrix} = \qquad (285)$$

This shows that $\perp$ to the field (assumed to be only in z-direction), there is a pressure $\frac{B_z^2}{8\pi}$, so that force in $\hat{x}$ & $\hat{y}$ directions are

$$-\partial_j M_{xj} = -\partial_x \frac{B_z^2}{8\pi} = -\nabla_x P_{mag} \qquad (286)$$

and

$$-\partial_j M_{yj} = -\partial_y \frac{B_z^2}{8\pi} = -\nabla_y P_{mag} \qquad (287)$$

but along the $\hat{z}$ direction

Force is: $\quad -\partial_j M_{zj} = +\nabla_z \frac{B_z^2}{4\pi}$ (288)

This corresponds to a force that increases in the direction of increasing B_z. This is a <u>tension force</u> that resists stretching much like a rubber band. Note that the pressure force $\perp B_z$ is in the direction of decreasing B_z^2, just like particle pressure force, whereas the tension force is in the direction of increasing B_z along the field line.

Having discussed the physical meaning of the terms in the momentum eqn let us consider some aspects of the magnetic induction equation:

$$\partial_t B = \nabla \times v \times B + \nu_m \nabla^2 \vec{B} \qquad (289)$$

First, note that the order of magnitude ratio of the 1st term on the right, to the second term on the right is given by →

$$R_M \equiv \frac{VB/L}{\nu B/L^2} = \frac{LV}{\nu_m} = \text{Magnetic Reynolds Number} \quad (240)$$

ν_m ← magnetic diffusivity

where V, L are characteristic velocities & scale of field variation in problem of interest.

(R_m is reminiscent of the the Reynolds number for hydrodynamic flows $\frac{LV}{\nu}$ ← (dynamic viscosity))

From (279) & (267)

$$\nu_m = \frac{5.5 \times 10^{11}}{T^{3/2}} \ln\Lambda \quad (291)$$

which is $\nu_m \simeq 10^7 \text{cm}^2/\text{s}$ for $T = 10^4$ K, $\ln\Lambda = 10$.

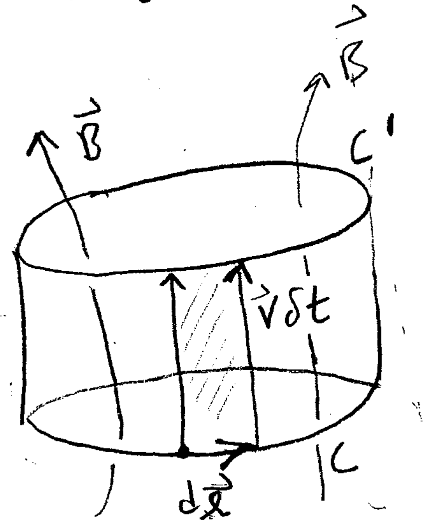
For a laboratory system, $L \simeq 10^2$ cm, $V \simeq 10$ cm/s

$\Rightarrow R_m \simeq 10^{-4}$.

For solar convection zone, $L \simeq 10^8$ cm, $V \simeq 10^5$ cm

$\Rightarrow R_m = 10^6$. Due to smaller scales and velocities involved, but temperatures that need not be hugely different, typically lab R_m is much smaller that astro R_m and usually R_m in astro $\gg 1$. →

For large R_m, the ν_m term can be ignored in the induction equation under most circumstances (but not all!). This leads to concept of flux freezing in astrophysics: (similar to Kelvin circulation theorem).



To prove: consider flux $\int \vec{B}\cdot d\vec{S}$ through closed contour C, moving with the fluid. Initial position at time t is closed contour C, and after time δt it has undergone displacement $\vec{v}\delta t$ to new position C'. Let $d\vec{S}_c$ be area element on C and $d\vec{S}'_{c'}$ be area element on C'. The area element with outward normal (shaded) is given by $d\vec{\ell} \times \vec{v}\delta t$.

Now $\nabla\cdot\vec{B} = 0$ implies that $\int \nabla\cdot B\, dV = \oint \vec{B}\cdot d\vec{S} = 0$. integrated around the closed cylinder. Thus

$$\int_c d\vec{S}_c \cdot \vec{B}(t+\delta t) - \int_{c'} d\vec{S}'_c \cdot \vec{B}(t+\delta t) - \int_C \vec{B}(t+\delta t)\cdot(d\vec{\ell}\times\vec{v}\delta t) = 0 \qquad (292)$$

Now:

$$\delta\Phi \equiv \int_{c'} d\vec{S}'_{c'} \cdot \vec{B}(t+\delta t) - \int_c d\vec{S}_c\, \vec{B}(t) \qquad (293)$$

which, using (292), becomes

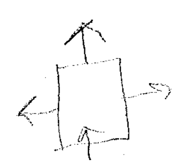
$$\delta\Phi \equiv \int_c d\vec{S}_i \cdot \vec{B}(t+\delta t) - \int_c \vec{B}(t+\delta t)\cdot(d\vec{l}\times\vec{v}\,\delta t) - \int_c d\vec{S}_i\cdot\vec{B}(t) \qquad (294)$$

$$= \delta t\left[\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}_c - \underbrace{\int \vec{B}(t+\delta t)\cdot(d\vec{l}\times\vec{v})}_{\simeq \int B(t)\cdot(d\vec{l}\times\vec{v}) \text{ for small } \delta t}\right]$$

$$\Rightarrow \quad = \delta t\left[\int \underbrace{\left(\vec{\nabla}\times(\vec{v}\times\vec{B}) + \nu_m \nabla^2\vec{B}\right)}_{\text{from } (279)}\cdot d\vec{S}_c - \int B(t)\cdot(d\vec{l}\times\vec{v})\right]$$

$$= \delta t\left[\left[\int(\vec{v}\times\vec{B})\cdot d\vec{l} + \int \nu_m \nabla^2\vec{B}\cdot d\vec{S}_c\right] - \int B(t)\cdot(d\vec{l}\times\vec{v})\right]$$

But $(\vec{v}\times\vec{B})\cdot d\vec{l} = \vec{B}\cdot d\vec{l}\times\vec{v}$ (vector identity)



so

$$\Rightarrow \quad \delta\Phi = \delta t \int \nu_m \nabla^2\vec{B}\cdot dS_c \qquad (295)$$

or $$\frac{d\Phi}{dt} = \int \nu_m \nabla^2\vec{B}\cdot dS_c \quad \Rightarrow \quad \text{for } R_m \gg 1$$

$$\frac{d\Phi}{dt} \simeq 0 \equiv \underline{\text{Flux freezing}}$$

$\Rightarrow$

Flux freezing is simply the statement that the magnetic field moves with the plasma so as to maintain $\int \vec{B}\cdot d\vec{S}$ = constant with time. If flux freezing were to apply during the collapse of a star like the sun, could it be a simple explanation for the origin of Neutron star magnetic fields? The sun has a mean field of order 2-10 Gauss. Flux freezing from $R_\odot \simeq 10^{11}$ cm to $R_{NS} \simeq 10^{6}$ cm implies an increase in field strength of order $\frac{R_{NS}^2}{R_\odot^2} \Rightarrow B_{NS} \lesssim 10^{11}$ Gauss.

Not bad. Many people believe this is possible, but others feel that young NS incur neutrino driven turbulent convection which can destroy the frozen in field with enhanced diffusion but also generate new field by dynamo action.

Magnetohydrostatics

As simple examples of MHD, consider time independent, velocity free equilibria:

$$f - \nabla p + \frac{1}{4\pi}(\nabla \times B) \times B = 0 \qquad (296)$$

Consider Body forces $= 0 \Rightarrow$

$$\nabla p = \frac{1}{4\pi}(\nabla \times B) \times B \qquad (297)$$

A magnetic field satisfying (297) is called a <u>pressure balanced</u> field.

An important dimensionless parameter is the plasma beta:

$$\beta \equiv \frac{p}{B^2/8\pi} \qquad (298)$$

Often in lab, $\beta \ll 1$. In astrophysics, the definition of "corona" for MHD people is often taken to be the region in stellar atmospheres or above accretion discs above which β drops below 1.

Note that when $\beta << 1$, (297) becomes

$$(\nabla \times B) \times B = 0 = \vec{J} \times \vec{B} \qquad (299)$$

This is called the <u>force-free</u> condition and implies that the magnetic pressure and tension forces conspire to balance. Note also that $J \times B = 0$ $\Rightarrow$ $J \parallel B$, so that $\vec{\nabla} \times \vec{B} \parallel \vec{B}$. (300)

Now consider an example of a pressure balanced column. We work in cylindrical coordinates, assuming cylindrical symmetry (no variation in θ, z).

Then from $\nabla \cdot B = 0$: $\frac{1}{r} \partial_r (B_r r) = 0$

or $B_r = \frac{\text{constant}}{r}$ but in order not to diverge at $r = 0$, constant must be zero. Thus $B_r = 0$.

We then write

$$\vec{B} = B_\theta(r)\hat{e}_\theta + B_z(r)\hat{e}_z \qquad (301)$$

$\rightarrow$

using 301 in 297 : =>

$$\frac{\partial p}{\partial r} = \frac{1}{4\pi}\left(-\frac{\partial B_z}{\partial r}\hat{e}_\varphi + \frac{1}{r}\frac{\partial}{\partial r}(r B_\varphi)\hat{e}_z\right) \times \vec{B}$$

$$= \frac{1}{4\pi}\left(-\frac{1}{2}\frac{\partial B_z^2}{\partial r} - \frac{1}{2}\frac{\partial (B_\varphi^2)}{\partial r} - \frac{B_\varphi^2}{r}\right)$$

$$\Rightarrow \frac{\partial}{\partial r}\left(p + \frac{B_\varphi^2}{8\pi} + \frac{B_z^2}{8\pi}\right) + \frac{B_\varphi^2}{4\pi r} = 0 \qquad (302)$$

assuming $p = p(r)$. Now consider that the magnetic field in the plasma column is produced by driving a current $j = j(r)\hat{e}_z$ along the axis of the column. This would only produce a field in the toroidal direction since

$$\frac{c}{4\pi}\nabla \times B = J_z \quad \text{and } B_r = 0 \Rightarrow \vec{B} = \vec{B}_\varphi$$

this relation is then

$$\frac{1}{r}\frac{d}{dr}(r B_\varphi) = \frac{4\pi J_z}{c} \qquad (303)$$

$\Rightarrow$

$\frac{V_A^3}{\lambda_\parallel} = \frac{V_\perp^3}{\lambda_\perp}$

$\frac{\lambda_\parallel}{V_A} =$

$\frac{1}{V_A} =$

$\frac{V_A^3}{\lambda_\parallel} = \frac{V_\perp^3}{\lambda_\perp}$

$\lambda_\perp = \frac{V_\perp^3}{V_A^3}\lambda_\parallel$

$\boxed{\frac{\lambda_\parallel}{V_A} = \frac{\lambda_\perp}{V_\perp}}$

$V_\perp = \frac{\lambda_\perp}{\lambda_\parallel} V_A$

$\frac{V_A^3}{\lambda_\parallel} = \left(\frac{\lambda_\perp}{\lambda_\parallel}\right)^3 \frac{V_A^3}{\lambda_\perp}$

$\frac{V_\perp^3}{\lambda_\perp} = \text{const}$

$\frac{V_A}{\lambda_\parallel} = \frac{V_\perp}{\lambda_\perp}$

$\frac{\lambda_\perp^2 V_A^3}{\lambda_\parallel^3} = \text{const}$

$\frac{k_\parallel^3}{V_A^3} = \text{const}$

If we now assume constant $j(r) = \bar{j}$, so that $\partial_r \bar{j} = 0$, then we can integrate (303):

$$\Rightarrow \quad \boxed{B_\varphi = \frac{2\pi j_z}{c} r} \qquad (304)$$

plugging into (302) =>

$$\frac{d}{dr}\left(p + \frac{4\cancel{\pi^2}}{c^2}\frac{j^2 r^2}{\cancel{8\pi}_{2}}\right) + \frac{j^2 r \pi}{c^2} = 0$$

or $\quad \frac{d}{dr} p = -\frac{2\pi}{c^2} r j^2$

$$\Rightarrow \quad \boxed{p = p_0 - \frac{\pi r^2}{c^2} j^2} \quad (\text{for constant } j) \qquad (305)$$

soln is plotted below: note that p drops as B_φ increases suggesting that pressure is concentrated by the "hoop pinch force" of the B_φ field.

=> MAGNETIC COLLIMATION OF JETS IN ASTRO!