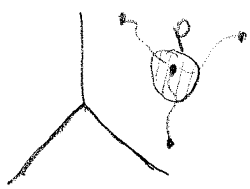
Turbulence (intro)

Once instabilities ensue linear theory of perturbations fail, non-linear theory is required.

Consider a point in phase space at an unstable equilibrium:



if P is unstable equilibrium, then small perturbation around around P sends the system off into an arbitrary direction of phase space. It then becomes impossible to predict the subsequent evolution phase space exactly. For fluid unstable to Kelvin-Helmholtz, Rayleigh-Taylor etc. The eventual consequence of the instability is turbulence. Random velocities every which way.

Though instabilities lead to turbulence, it is also possible to produce them with random stirring of a fluid at different locations.

No deterministic theory of turbulence is possible, but one can develop a theory of average properties. What kind of "average"?

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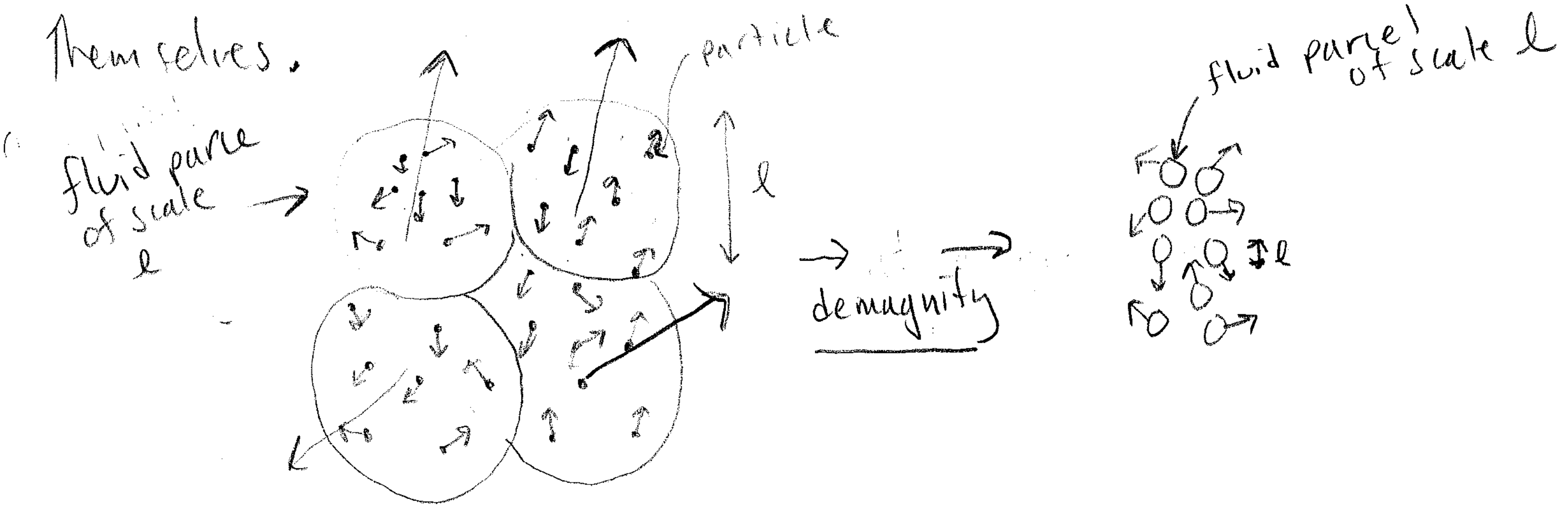
- Volume average – good for spatially homogeneous or nearly spatially homogeneous
- time average – good for temporally steady or nearly steady
- ensemble average: average over many hypothetical copies of the system having same statistical properties, but which differ in the actual values of quantities like velocity at a given space and given time

when the variation time or spatial gradient scales are large compared to fluctuating scales, then ensemble average and volume or time averages can be thought to be equivalent.

Note the difference between the averaging over fluid velocity fluctuations and the averaging over kinetic theory! to get the fluid eqns. themselves.



fluid
Velocity at given location can be written

$v = \bar{v} + v'$, where $\bar{v}$ is mean & v' is fluctuation. By construction $\overline{v'} = 0$.

Consider the statistical quantity $\overline{v'(x,t) \cdot v'(x+r,t)}$. If $\vec{r} = 0$, then this is $\overline{v'^2}$ which is a measure of kinetic energy in the turbulence. But if $\vec{r}$ is large then $\overline{v'(x,t) \cdot v'(x+r,t)} = 0$. Thus such a <u>correlation</u> has sizeable values only within a range Δr. This is the correlation length. Such correlations contain information about the strength & correlation length of the turbulence. A statistical theory of turbulence is one that develops equations for these correlations. Higher correlations are also often needed, e.g. the 3-point correlation $\overline{v_i(\vec{x})\, v_j(\vec{x}_2)\, v_k(\vec{x}_3)}$.

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Even statistical theories of turbulence involve many approximations: Closure problem: differential equations for n-point correlations depend on $(n+1)$-point correlations and in general an approximation is needed to close the equations.

Even a simple problem like convectively unstable fluid heated from below with top and bottom temperatures given does not have known rigorous solution for $\overline{V_i(\vec{x})\,V_j(\vec{x}+\vec{r})}$.

The statement that "turbulence is an unsolved problem in physics" means that we do not yet understand how to calculate n-point correlations from a fundamental theory.

Kinematics of Homogeneous Isotropic Incompressible Turbulence:

In this simple limit we can derive some properties of turbulence from symmetry:

First, note that a mean flow violates isotropy so we set it to zero and thus $\vec{V} = \overline{\vec{V}} + \vec{V}' = \vec{V}'$.

Second, homogenity requires $\overline{V_i(x)\,V_j(x+\vec{r})}$ is independent of $\vec{x}$, depending only on $\vec{r}$. Thus we write

$$\overline{V_i(x)\,V_j(\vec{x}+\vec{r})} = R_{ij}(|r|)$$

$\rightarrow$

Then $$\frac{\partial R_{ij}}{\partial r_j} = \overline{v_i(x)\,\frac{\partial v_j(\vec{x}+\vec{r})}{\partial r_j}} = 0$$

(assuming incompressible $\nabla \cdot v = 0$). Since R_{ij} depends only on $|\vec{r}|$, $R_{ij} = R_{ji}$

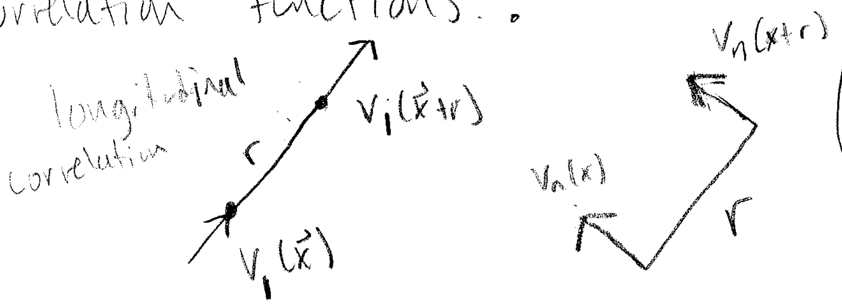
(that is, $\overline{v_i(\vec{x})\,v_j(\vec{x}+\vec{r})} = \overline{v_j(x)\,v_i(x+\vec{r})}$),

then $$\frac{\partial R_{ij}}{\partial r_j} = \frac{\partial R_{ji}}{\partial r_j} = 0.$$

Von Kármán & Howarth (1938) showed that the most general tensor function $R_{ij}(r)$ is then

$$R_{ij}(r) = A(r)\,r_i r_j + B(r)\,\delta_{ij} \qquad (170)$$

Consider longitudinal and lateral (also "normal") velocity correlation functions:



(normal or lateral correlation)

Since longitudinal component of $\vec{r}$ is $r_\ell = r$ and normal component of $\vec{r}$ is $r_n = 0$, we have

$$R_{\ell\ell}(r) = A(r)\,r^2 + B(r) = \tfrac{1}{3}\overline{v^2} f(r) \qquad (171)$$

$$R_{nn}(r) = B(r) = \tfrac{1}{3}\overline{v^2}\,(g(r)) \qquad (172)$$

$f(r)$ & $g(r)$ are defined such that $f(0) = g(0) = 1$. (at $r=0$).

We can then express $A(r)$, $B(r)$ in (170) using (171), (172), in terms of $f(r)$, $g(r)$:

$$R_{ij} = \frac{1}{3}\bar{v}^2\left[\frac{f(r)-g(r)}{r^2} r_i r_j + g(r)\delta_{ij}\right]$$

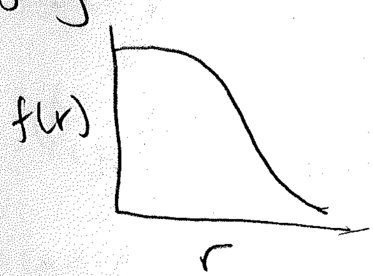
then using $\dfrac{\partial R_{ij}}{\partial r_i} = \dfrac{\partial R_{ij}}{\partial r_j} = 0 \Rightarrow$ use

$$\frac{\partial_i (f-g) r_i r_j}{r^2} - \frac{2r \frac{dr}{dr_i} r_i r_j}{r^4}(f-g) + \frac{(f-g)}{r^2}(3r_j + \delta_{ij} r_i) + \partial_i g \delta_{ij}$$

and $\dfrac{dr}{dr_i} = \dfrac{r_i}{r}$ and multiply by r_j

$$g(r) = f(r) + \frac{1}{2} r \frac{df}{dr} \qquad (173)$$

Thus if we can determine $f(r)$, we can get all components of the correlation tensor R_{ij}. Since $f(r)$ is the longitudinal correlation function, we expect it to have a decaying form



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Consider the fourier transform of

R_{ij} : $\phi_{ij}(\vec{k}) = \frac{1}{(2\pi)^3}\int R_{ij}(r) e^{-i\vec{k}\cdot\vec{r}} d^3r$

Since R_{ij} is spherically symmetric in $\vec{r}$,

ϕ_{ij} must be spherically symmetric in $\vec{k}$

so write $\phi(\vec{k}) = \phi(k)$. note "+" sign

Then

$$R_{ij}(r) = \int \phi_{ij}(k) e^{+i\vec{k}\cdot\vec{r}} d^3k \qquad (174)$$

Incompressibility $\frac{\partial R_{ij}}{\partial r_j} = \frac{\partial R_{ji}}{\partial r_j} = 0$ requires; at each k

$$k_i \phi_{ij} = k_j \phi_{ij} = 0$$

Symmetry considerations then require (e.g. McComb "Physics of Fluid Turbulence").

$$\phi_{ij}(k) = C(k) k_i k_j + D(k)\delta_{ij} \quad , \text{ where}$$

$$D(k) = -C(k)k^2 ,$$

(defines $E(k)$)

then

$$\phi_{ij}(k) = \frac{E(k)}{4\pi k^4}\left(k^2\delta_{ij} - k_i k_j\right) \qquad (175)$$

Significance is that

$$\frac{1}{2}\overline{v^2} = \frac{1}{2}R_{ii}(0) = \frac{1}{2}\int \phi_{ii}(k)\, d^3k \qquad (176)$$

using (174).

But using (175) in (176) and writing

$d^3k = 4\pi k^2 dk$ gives

$$\frac{1}{2}\overline{v^2} = \int_0^\infty E(k)\,dk \qquad (177)$$

Thus $E(k)$ is the <u>energy spectrum</u> of the turbulence. Just as B-body spectrum is composed of contributions at different wavelengths, turbulence can be thought of as being composed of contributions of different Fourier components.

Note that $f(r)$ and $E(k)$ are unspecified and are related such that only one is independent. We have not said anything about the form of $E(k)$, that is where <u>Kolmogorov theory</u> fits in

→

Kolmogorov Equilibrium Theory

A turbulent fluid can be maintained in a steady-state only if energy is continuously fed into system.
Reason is viscous dissipation → left alone, turbulent energy will convert to heat.
If the fluid is stirred, such that turbulent flow is statistically homogeneous & isotropic, then steady state can ensue, and system is in statistical equilibrium. Kolmogorov (1941) calculated the energy spectrum for such turbulence.

Imagine the driving (or forcing) to occur on some scale ℓ, inducing velocity v. Kolmogorov intuited that the the turbulent parcels (or eddies) of would feed energy to smaller scales, which then feed energy to still smaller scales. To see how this cascade proceeds, consider incompressible turbulence.

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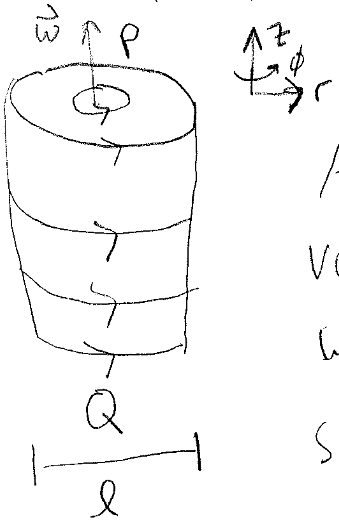
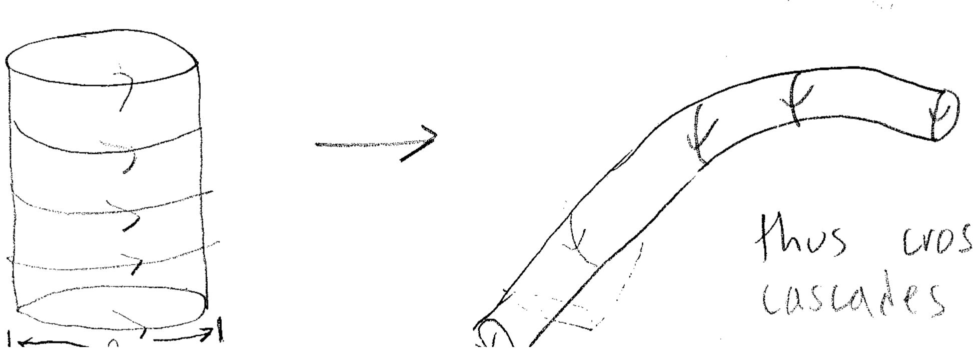
Let P & Q be two fluid elements on a vortex tube as shown, with diameter l:



According to Kelvin's theorem vorticity is conserved, or carried with the flow, (e.g. $\frac{d}{dt}\int w\cdot d\vec{s} = 0$ so that $\int \vec{w}\cdot d\vec{s}$ = constant, as derived earlier)

Then, since, statistically speaking, two points in the turbulent flow tend to separate with time, the vortex tube will lengthen as the points separate, still maintaining the coherence of the tube. But incompressibility requires that the the lengthened tube contract: Fixed density means fixed mass for given volume, stretching the tube in length requires decreasing the cross section to maintain the same density for the same mass of material. Thus



$s \ll l$

thus cross section cascades in scale →

the continuous shrinking of the vortex tubes cannot continue forever because eventually viscous term $\nu \nabla^2 V$ becomes important in the Navier-Stokes equation. That is the conditions for

viscosity

Kelvin's theorem are violated for small enough scales (large enough gradients), and vorticity is dissipated. Another way of saying this is that the Reynolds number for the smallest eddies is of order ~ 1 : $R_d \simeq \frac{l_d V_d}{\nu} \simeq 1$ whereas $R_0 \simeq \frac{LV}{\nu} \gg 1$, where subscript d refers to dissipation scale, and R_0 is the Reynolds number for the largest scale (or "outer" scale) of length L & velocity V.

So the idea is that vortex energy is input at scale L with velocity V; it then cascades to scale l_d where it is dissipated into random particle energy (heat).

In steady state energy input rate must equal energy dissipation rate →

and this energy transfer rate will be the same at all scales in a steady state. per unit mass, dimensional analysis leads to

$$\frac{dE}{dt} \sim \frac{V_\ell^3}{\ell} \quad \text{at scale } \ell \text{ with velocity } V_\ell$$

since $$\frac{V_\ell^3}{\ell} = \frac{V^3}{L} = \frac{V_d^3}{\ell_d} \qquad (177)$$

and $V_d \, \ell_d \sim \nu$ we have

$$\frac{V^3}{L} \simeq \frac{\nu^3}{\ell_d^4} \Rightarrow R^3 = \frac{L^4}{\ell_d^4} \quad \text{or} \quad \frac{L}{\ell_d} = R^{3/4} \qquad (178)$$

thus the Reynolds number determines the ratio of largest to smallest scales in the cascade, this range of scales is called the <u>inertial range.</u>

To get the energy spectrum, one just uses $\frac{V_\ell^3}{\ell} =$ constant

$$\Rightarrow \quad V_\ell \propto \ell^{1/3} \Rightarrow V_\ell^2 \propto \ell^{2/3} \quad \text{and} \quad \ell \sim k_\ell^{-1}$$

thus $V_\ell^2 \propto k^{-2/3}$ $\rightarrow$

now kinetic energy density per mass v^2 around wavenumber k is $E(k)\,dk \sim E(k)\,k$

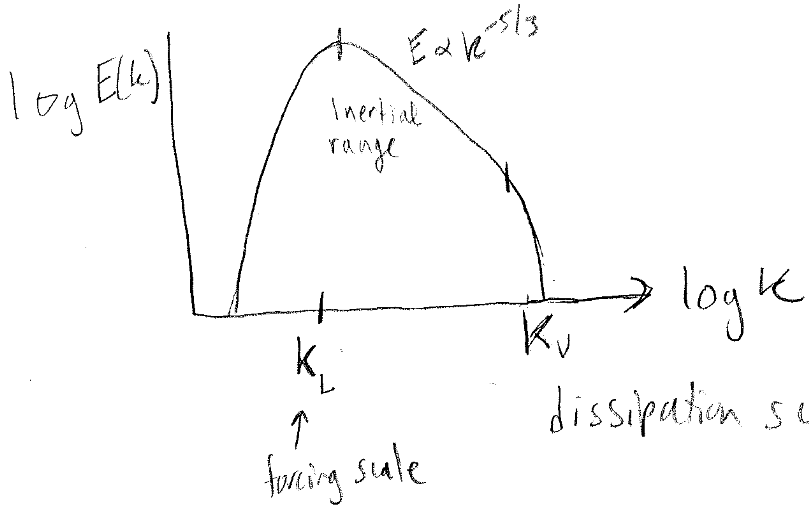
for $dk \sim k$

then $v_\ell^2 = E(k)k \propto k^{-2/3}$

$$\Rightarrow \quad E(k) \propto k^{-5/3} \qquad (179)$$

this is the Kolmogorov spectrum.

It applies for the inertial range:



Turbulent Diffusion (part 1)

Though homogeneous isotropic turbulence is simplest, real systems have inhomogenities. Turbulence affects transport, and the simplest effect is turbulent diffusion:

If you put sugar in coffee and do not stir, mixing occurs by molecular processes and takes a long time. But if stirred, the coffee becomes turbulent and mixing occurs more quickly.

Suppose markers are introduced in fluid at $t=0$. Displacement of marker after time $t=T$ is

$$\vec{X}(T) = \int_0^T \vec{V}_L(t)\,dt \tag{180}$$

where $V_L(t)$ is fluid velocity at time t. Subscript L indicates Lagrangian approach (following marker in the fluid). The mean displacement averaged over all markers must vanish, for a volume fixed in space (e.g. the coffee cup) but the mean squared displacement does not vanish:

$$\overline{X^2(T)} = \int_0^T dt \int_0^T dt \; \overline{\vec{V}_L(t) \cdot \vec{V}_L(\tau)} \qquad (181)$$

where $\overline{\vec{V}_L(t) \cdot \vec{V}_L(\tau)}$ is the velocity correlation function for velocities at two different times, but at fixed position. In steady state this must depend only on $t-\tau$, so we write

$$\overline{\vec{V}_L(t)\vec{V}_L(\tau)} = \overline{V^2} R(t-\tau) \qquad (182)$$

note $R(0) = 1$. We also assume symmetry: $R(t-\tau) = R(\tau-t)$. We expect turbulence to have some correlation time τ_{cor} such that $R(\tau)$ is only substantially finite of $\tau < \tau_{cor}$ (τ_{cor} is typically an eddy turnover time, $\frac{l_{cor}}{V}$)

using (182) in (181) $\Rightarrow$

$$\overline{X^2(T)} = \int_0^T dt \, \overline{V^2} \int_0^T d\tau \, R(\tau - t) \qquad (183)$$

Consider $T \ll \tau_{cor}$: then $R(\tau - t) \simeq 1$, so

$$\overline{X^2(T)} = \bar{V}^2 T^2 \qquad (184)$$

as expected. But for $T \gg \tau_{cor}$ statistical effects of turbulence emerge:

In this limit, we can change integration bounds to $-\infty$ and $+\infty$ (since away from τ_{cor} there is little contribution)

$$\overline{X}^2(T) = \int_0^T dt\, \bar{V}^2 \int_{-\infty}^{\infty} d\tau\, R(\tau - t) \qquad (185)$$

we can choose $t = 0$

Then writing $D_T = \frac{1}{3} \bar{V}^2 \int_0^\infty R(\tau)\, d\tau$

and assuming $\bar{V}^2$ is independent of time & space

$$\overline{X}^2(T) = 6 D_T T \qquad (186)$$

(where $6 = 2 \cdot 3$ and the 2 comes from $\int_{-\infty}^{\infty} \to 2\int_0^\infty$)

Now we argue that D_T can be thought of as a diffusion coefficient

$\longrightarrow$