Exam 2 Solutions (AST 231 2008 Blackman)

$$ds^2 = -(1-\tfrac{2M}{r})dt^2 + (1-\tfrac{2M}{r})^{-1}dr^2 + r^2\sin\theta\, d\phi^2 + r^2 d\theta^2$$

Q1) at closest appraoch $r=R$, the turning point, there $\left.\frac{dr}{d\tau}\right|_R = 0$. From 9.21, 9.22:

(A) $$\frac{d\phi}{d\tau} = \frac{\ell}{r^2} \quad (\text{for } \sin\theta = \tfrac{\pi}{2})$$

- locally, SR holds so energy measured by observer is

$$E = \frac{m_a}{(1-v^2)^{1/2}} = -p \cdot u_{obs} \qquad (1)$$

m_a ← asteroid mass

p: 4-velocity of asteroid

$-p\cdot u_{obs}$: Energy measured by observer with 4-velocity u_{obs}

- for stationary observer at R

$$g_{\mu\nu} u^\mu_{obs} u^\nu_{obs} = -1 = g_{00} {u^t_{obs}}^2 = (1-\tfrac{2M}{R}) {u^0_{obs}}^2$$

$$\Rightarrow u^t_{obs} = (1-\tfrac{2M}{R})^{+1/2}$$

$$u^\mu_{obs} = ((1-\tfrac{2M}{R})^{-1/2}, 0, 0, 0)$$

→

Black hole mass

$$\Rightarrow \quad E = -g_{tt} p^t u^t_{obs} = p^t \left(1 - \frac{2M}{R}\right)^{1/2}$$

$$= m_a u^t \left(1 - \frac{2M}{R}\right)^{1/2} \qquad (2)$$

asteroid mass

Since $u \cdot u = -1 = g_{\mu\nu} u^\mu u^\nu$

we can obtain u^t: (at $r = R$, have only u^t & u^ϕ)
use $\sin\theta = \frac{\pi}{2}$

$$-\left(1 - \frac{2M}{R}\right) u^{t2} + R^2 u^{\phi 2} = -1$$

but $u^\phi = \frac{d\phi}{d\tau} = \frac{\ell}{r^2}$ from (A) so at $r = R$

$$\Rightarrow -\left(1 - \frac{2M}{r}\right) u^{t2} + \cancel{R^2} \frac{\ell^2}{R^{\cancel{4}2}} = -1$$

[from (1) & (2)]

$$\Rightarrow \quad u^t = \frac{\left(1 + \frac{\ell^2}{R^2}\right)^{1/2}}{\left(1 - \frac{2M}{R}\right)^{1/2}} \Rightarrow V = \frac{\ell/R}{\left[1 + (\ell/R)^2\right]^{1/2}} \qquad (3)$$

- need to get $\ell(b, R)$
- at large $r \gg R$, impact parameter $b = r\phi$

so there $\ell \equiv r^2 \frac{d\phi}{d\tau} = r^2 \frac{d\phi}{dr}\frac{dr}{d\tau} = r^2\left(\frac{-b}{\cancel{r^2}}\right)\frac{dr}{d\tau} = -b\frac{dr}{d\tau} \qquad (4)$

But at large r, V_{eff} in eq 9.29 vanishes

so $\mathcal{E} = \frac{e^2 - 1}{2} = \left(\frac{dr}{d\tau}\right)^2 / 2 \Rightarrow \frac{dr}{d\tau} = (e^2 - 1)^{1/2}$ at large r.

This relates ℓ to e in (4): $\Rightarrow$

$$\ell = +b(e^2 - 1)^{1/2} \qquad (5)$$

$\uparrow$ since $\frac{dr}{d\tau} < 0$.

$\rightarrow$

now turning point condition from equation 9.29 : $\frac{dr}{d\tau}=0$ at $r=R$ so from 9.28: (3)

$$\mathcal{E}=\frac{e^2-1}{2}=V_{eff}(R)=\frac{1}{2}\left[\left(1-\frac{2M}{R}\right)\left(1+\frac{\ell^2}{R^2}\right)-1\right]$$

$$e^2-1=\left(1-\frac{2M}{R}\right)\left(1+\frac{\ell^2}{R^2}\right)-1 \quad \text{at turning point}$$

$$=\frac{\ell^2}{R^2}-\frac{2M}{R}-\frac{2M\ell^2}{R^3} \qquad (6)$$

Solving (6) & (5) for ℓ^2/R^2

$$\Rightarrow \frac{\ell^2}{R^2}=\frac{b^2}{R^2}\left(\frac{\ell^2}{R^2}-\frac{2M}{R}-\frac{2M\ell^2}{R^3}\right)=\frac{\ell^2}{R^2}\left(\frac{b^2}{R^2}-\frac{2Mb^2}{R^3}\right)-\frac{2Mb^2}{R^3}$$

$$\Rightarrow \frac{\ell^2}{R^2}=\frac{2Mb^2/R^3}{\frac{b^2}{R^2}-\frac{2Mb^2}{R^3}-1}$$ now plug into (3):

$$\Rightarrow V^2=\frac{1}{(R/\ell)^2+1}=\frac{2Mb^2/R^3}{\frac{b^2}{R^2}-\cancel{\frac{2Mb^2}{R^3}}-1+\cancel{2Mb^2/R^3}}$$

$$\Rightarrow \boxed{V=\frac{\left(\frac{2M}{R}\right)^{1/2}\frac{b}{R}}{\left(\frac{b^2}{R^2}-1\right)^{1/2}}}$$

Q2) $\quad u \cdot u = g_{\alpha\beta} u^\alpha u^\beta$

$$\frac{d(u \cdot u)}{d\tau} = g_{\alpha\beta} u^\alpha \frac{du^\beta}{d\tau} + g_{\alpha\beta} \frac{du^\alpha}{d\tau} u^\beta + \frac{dg_{\alpha\beta}}{d\tau} u^\alpha u^\beta$$

$$= 2 g_{\alpha\beta} u^\alpha \frac{du^\beta}{d\tau} + \overbrace{\frac{\partial g_{\alpha\beta}}{\partial x^\mu} \underbrace{\frac{dx^\mu}{d\tau}}_{u^\mu}}^{\frac{dg_{\alpha\beta}}{d\tau} \text{ via chain rule}} u^\alpha u^\beta$$

$$= 2 g_{\alpha\beta} u^\alpha \frac{du^\beta}{d\tau} + \frac{\partial g_{\alpha\beta}}{\partial x^\mu} u^\mu u^\alpha u^\beta$$

geodesic equation 8.15 : $\quad \frac{du^\beta}{d\tau} = -\Gamma^\beta_{\gamma\sigma} u^\gamma u^\sigma$

$\Rightarrow$

$$= -2 g_{\alpha\beta} u^\alpha \Gamma^\beta_{\gamma\sigma} u^\gamma u^\sigma + \frac{\partial g_{\alpha\beta}}{\partial x^\mu} u^\mu u^\alpha u^\beta$$

$$= u^\gamma u^\sigma \left(-2 g_{\alpha\beta} u^\alpha \Gamma^\beta_{\gamma\sigma} + \frac{\partial g_{\gamma\sigma}}{\partial x^\mu} u^\mu \right)$$

from 8.19 : $\quad g_{\alpha\beta} \Gamma^\beta_{\gamma\sigma} = \frac{1}{2} \left(\frac{\partial g_{\alpha\gamma}}{\partial x^\sigma} + \frac{\partial g_{\alpha\sigma}}{\partial x^\gamma} - \frac{\partial g_{\gamma\sigma}}{\partial x^\alpha} \right)$

$$\Rightarrow u^\gamma u^\sigma \left(-u^\alpha \frac{\partial g_{\alpha\gamma}}{\partial x^\sigma} + u^\alpha \frac{\partial g_{\alpha\sigma}}{\partial x^\gamma} \underbrace{\cancel{- u^\alpha \frac{\partial g_{\gamma\sigma}}{\partial x^\alpha}} + \cancel{\frac{\partial g_{\gamma\sigma}}{\partial x^\mu} u^\mu}}_{\text{cancel}} \right)$$

$$= \underbrace{\cancel{-u^\gamma u^\sigma u^\alpha \frac{\partial g_{\alpha\gamma}}{\partial x^\sigma}} + \cancel{u^\gamma u^\sigma u^\alpha \frac{\partial g_{\alpha\sigma}}{\partial x^\gamma}}}_{\text{(cancel)}}$$

$= \boxed{0}$ //

(remember: repeated indices sum & $g_{\alpha\beta}$ is symmetric)

Q3) $-dt^2 + a^2(t)(dx^2+dy^2+dz^2)$

metric coefficients are independent of x, y, z. 3-killing vectors corresponding to each of these directions.

Consider propagation in x direction; associated killing vector is $\underset{\sim}{\xi}^{\mu}_{x} = (0,1,0,0)$.

Let $\underset{\sim}{p}$ be photon 4-momentum.

Then $\underset{\sim}{\xi}_x \cdot \underset{\sim}{p}$ is conserved.: so

$$\underset{\sim}{\xi}_x \cdot \underset{\sim}{p} = g_{\mu\nu}\, \xi^{\mu}_{x} p^{\nu} = g_{xx} P^{x}(t) = a^2(t)\, p^{x}(t) = \text{constant}$$

$$\Rightarrow p^x \propto \frac{1}{a^2(t)} \quad (1)$$

since $\underset{\sim}{p}$ is null:

$$\underset{\sim}{p}\cdot\underset{\sim}{p} = 0 = -p^{t^2} + a^2(t)\, p^{x^2} = 0$$

(from (1))

$$\Rightarrow p^t(t) = a(t)\, p^x \Rightarrow p^t \propto \frac{1}{a(t)} \quad (2)$$

$\therefore$ stationary observer has $U_{obs} = (1,0,0,0)$ in this metric and measures energy

$$E = -p \cdot u_{obs} = -g_{tt} p^t = +p^t(t) \propto \frac{1}{a(t)} \quad \text{from (2).}$$

$$\Rightarrow \frac{E(t)}{E(t_0)} = \frac{\omega(t)}{\omega(t_0)} = \frac{a(t_0)}{a(t)} = \text{cosmological redshift}$$

(Q4) from box 11.1d or e

$$4\pi R^2 \sigma T^4 = \varepsilon L_{edd}$$

$$\Rightarrow T = 5\times10^7 \left(\frac{GM}{c^2 R}\right)^{1/2} \left(\varepsilon \frac{M_\odot}{m}\right)^{1/4} \text{ K}$$

$$W = 6\times10^{10}\, T \text{ (Hz)}$$

$$\Rightarrow \underline{W = 3\times10^{18} \left(\frac{GM}{c^2 R}\right)^{1/2} \left(\varepsilon \frac{M_\odot}{m}\right)^{1/4} \text{ Hz}}$$

for $M = 10^9 M_\odot$, $R = R_{isco} = 6\frac{GM}{c^2}$, $\varepsilon = 0.1$

$$\Rightarrow W = 3\times10^{18} \left(\frac{1}{6}\right)^{1/2} (0.1)^{1/4} (10^{-9})^{1/4} \text{ Hz}$$

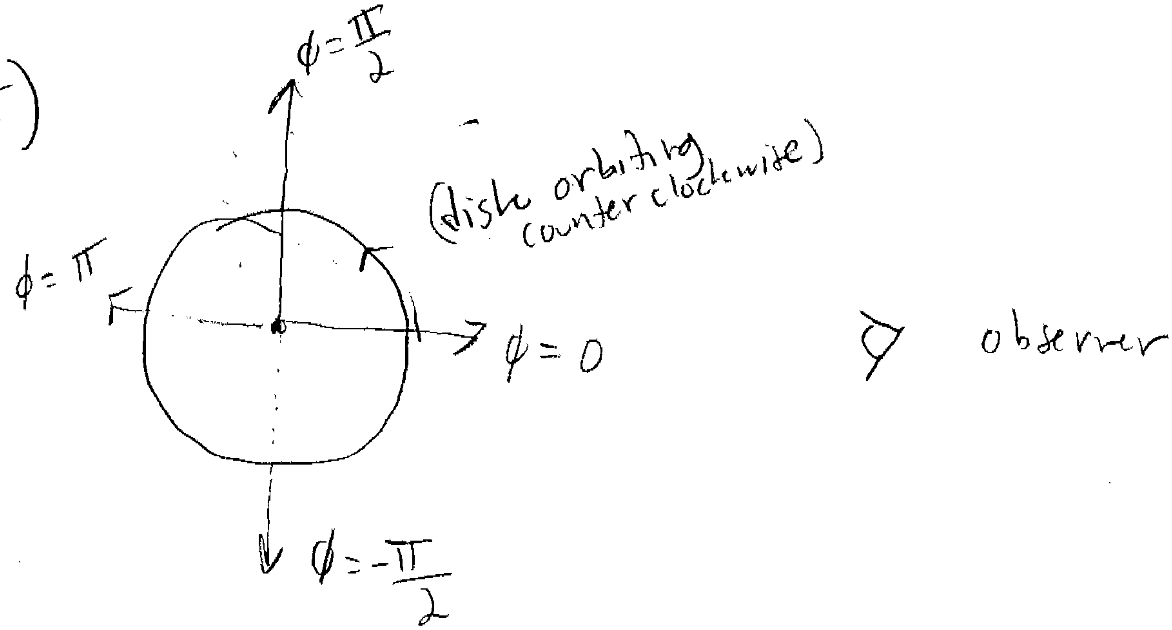
$$= \underline{3\times10^{15} \text{ Hz}} \quad \underline{\text{(ultraviolet)}}$$

- for $M = 10 M_\odot$, $\varepsilon = 0.1$, $R = R_{isco} = 6\frac{GM}{c^2}$:

$$\Rightarrow W = (3\times10^{15})\left(\frac{10^9}{10}\right)^{1/4} = \underline{3\times10^{17} \text{ Hz}}$$

(x-rays)

Q5)



- at $\phi = 0$ photons have no motion along line of sight so there is only gravitational redshift. At $\phi = \frac{\pi}{2}$, disk is moving <u>AWAY</u> from observer so the gravitational redshift and doppler shift both act to reduce the frequency. $\phi = \frac{\pi}{2}$ corresponds to minimum.

At $\phi = -\frac{\pi}{2}$ doppler shift increases frequency competing with the effect of grav. redshift. $\phi = -\frac{\pi}{2}$ would thus produce largest possible frequency observed at ∞

$\rightarrow$

From Eqn. 11-24 for $|\phi| = \frac{\pi}{2}$ ⑧

$$\frac{w_\infty}{w_{em}} = \frac{\left(1 - \frac{3M}{r}\right)^{1/2}}{\left(1 \pm \frac{1}{\left(\frac{r}{M} - 2\right)^{1/2}}\right)}$$

the minus sign must correspond to $\phi = -\frac{\pi}{2}$ since it gives the largest value.

at $r = 6M$

$$\frac{w_\infty}{w_{em}} = \frac{\left(1 - 1/2\right)^{1/2}}{\left(1 - \frac{1}{2}\right)} = \frac{2}{\sqrt{2}} = \boxed{\sqrt{2}}$$

b) Since $\sqrt{2} > 1 \Rightarrow$ Doppler effect dominates the gravitational redshift effect; Doppler effect of $\phi = -\frac{\pi}{2}$ acts to increase observed frequency from that emitted while grav. redshift acts to lower frequency. $\frac{w_\infty}{w_{em}} > 1$ means Doppler wins.