Quantum Entanglement, EPR paradox, and Bell's Inequality

Kapitza Society final paper

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Abstract

This paper is a summary of what we covered in one of our weekly discussions on quantum entanglement. It's largely a reproduction of section 4.1, 4.2 of Preskill's Intro to Quantum Computing lecture notes [1] and section 12.1, 12.2 of Griffth's Introduction to Quantum Mechanics [2].

1. Introduction

In quantum mechanics, particle behaves quite differently compared to the classical point of view. One of the mysterious phenomena that quantum mechanics predicted, and that experiments confirmed, is entanglement. In the early years when scientists developed quantum mechanics, it's indeed disturbing to accept the instant action and nonseparability between particles, considering Dr. Einstein just refreshed our mind with spacetime locality in couple of years ago. What's the property of entangled states? How to make measurements in entangled states? We will go through the technique and the science history development behind it.

2. The Nonseparability of Entangled States

We've talked about the "separable" states in previous meetings. We say the composite Hilbert space of single Hilbert space H_A and H_B, H_{AB}, is the tensor product of two space: $H_{AB} = H_A \otimes H_B$. The composite state is then the tensor product of two separate states from two subspace:

$$|\phi\rangle_{AB} = |\phi\rangle_A \otimes |\phi\rangle_B$$

In this case, making measurements on subsystem A or B doesn't affect the other subsystem. I might call this "separable pair." When say two qubits are "entangled", the two qubits subsystem is in a state of the following form:

$$|\phi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|11\rangle_{AB}$$

which is a linear combination of two orthonormal basis states with $\alpha^2 + \beta^2 = 1$. One can no longer measure one subsystem without affecting the other subsystem, because the state will collapse into the basis. The maximally entangled state is when $\alpha = \beta = \frac{1}{\sqrt{2}}$, in which when one makes a measurement, the state $|\phi\rangle$ has a fifty-fifty probability to collapse two basis. Correspondingly, the density operator will be:

$$\rho_A = tr_B(|\phi^+\rangle\langle\phi^+|) = \frac{1}{2}I_A$$

$$\rho_B = tr_A(|\phi^+\rangle\langle\phi^+|) = \frac{1}{2}I_B$$

This means the local measurement result will be completely random. The information of the state is likely "hidden" from us. One might construct a basis using, all 4 of the maximally entangled states in two qubits system:

$$|\phi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle_{AB} \pm \beta|11\rangle_{AB})$$

$$|\psi^\pm\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} \pm \beta|10\rangle_{AB})$$

Preskill used the example of Alice, Bob, and Charlie again. Suppose Charlie encodes one of the above state on two qubits A and B, sending A to Alice and B to Bob. If they can't communicate, none of them are able to figure out the "parity bit" $(+ \text{ or } -)$ and "phase bit" $(\phi \text{ or } \psi)$. However, locally, Alice or Bob may apply σ_3, σ_1 on her qubit.

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We have $|0\rangle_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A, |0\rangle_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B$, then

$$\sigma_3 |0\rangle_A = |0\rangle_A, \sigma_3 |1\rangle_A = -|1\rangle_A$$

$$\sigma_1 |0\rangle_A = |1\rangle_A, \sigma_1 |1\rangle_A = |0\rangle_A$$

Note that the entangled state will correspondingly transform. After applying σ_3, the phase bit will flip:

$$|\phi^+\rangle \leftrightarrow |\phi^-\rangle,\ |\psi^+\rangle \leftrightarrow |\psi^-\rangle$$

And after applying σ_1, the parity bit will flip:

$$|\phi^+\rangle \leftrightarrow |\psi^+\rangle,\ |\phi^-\rangle \leftrightarrow -|\psi^-\rangle$$

Once Alice and Bob communicate, they are able to determine which of the four states their qubits are in. One may concretely summarize the property as: the entangled basis states are the simultaneous eigenstates of two commuting observables (σ_3 and σ_1):

$$\sigma_1^{(A)} \otimes \sigma_1^{(B)}, \ \ \sigma_3^{(A)} \otimes \sigma_3^{(B)}$$

The moral of the story is that for such entangled qubits, we are allowed to manipulate the bits by applying certain operators, as long as they don't deteriorate the state. We don't need to know the information of the single qubit and that "separated" basis$\{|00\rangle_{AB}, |11\rangle_{AB}, |01\rangle_{AB}, |10\rangle_{AB}\}$, and rather we can just focus on the new basis of entangled states $\{|\phi^{\pm}\rangle, |\psi^{\pm}\rangle\}$. The qubit pairs are inseparable, so let's just stop treating them like independent object and measure them separately. I personally believe that this process inherits the classic technique of coordinate transformation, like what we do in degenerate perturbation theory (not sure if this is an abuse of math). One can also transform inversely. As Preskill points out, when being brought together, Alice and Bob may transform their pairs jointly, by applying a unitary transformation and rotate the basis $\{|\phi^{\pm}\rangle, |\psi^{\pm}\rangle\}$ back to $\{|00\rangle_{AB}, |11\rangle_{AB}, |01\rangle_{AB}, |10\rangle_{AB}\}$. Quantum circuits are actually based on these operators (Hadamard transform; CNOT transform).

3. Einstein Locality and EPR

The Einstein locality may be summarized as:

> Suppose that A and B are spacelike separated systems. Then in a complete description of physical reality an action performed on system A must not modify the description of system B.

Indeed, Einstein was discomforted with the entanglement phenomenon. The entanglement phenomenon implies instant actions that exceeds the speed of light, which violate the Einstein locality. In 1953, He along with Podolsky and Rosen (EPR) brought up this as the EPR paradox (The two particle pairs, two qubits pairs, are all called EPR pairs). In fact, he would not like to accept the indeterminacy of quantum theory at all and claimed that the theory is "incomplete." The "determinant" theories, which admired by Einstein, are also called local hidden-variable theories. Not like quantum theory that adopts a totally probabilistic interpretation of the world, hidden-variable theories believe that the world appears to be probabilistic because we are not omnipotent to know everything, i.e., there're some hidden variables, for example, λ. Einstein believed that the quantum theory needs to include some more hidden variables to be complete and to reveal the fundamentally deterministic viewpoint of the world.

Griffiths in his textbook points out why the paradox could be so disturbing. Physicists never wish to abandon the Einstein locality, but they have no choice when they notice that the conservation of momentum may collapse, an even more horrible consequence. Consider the creation of electron-positron pair in a π^0 decay reaction. The electron and positron will fly away from each other, and they occupy the singlet spin state:

$$\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$

As quantum physics predicts, and as what experiments showed, measuring one of them and find it to be in a spin-up state, we immediately know that the other is in the spin-down state; vice versa, no matter how far they are. By locality, we might propose that it takes some time for the entanglement action to "travel" in the space. That's when the conservation of moment collapses. Thus we say in entanglement, the collapse of the wave function is instantaneous.

4. The Bell Inequality

Bell inequality proved that none of the hidden-variable theories will save us from the paradox. Preskill used Alice and Bob's dialogue example; I personally prefer Bell's thought experiment that Griffiths presents in his textbook. Suppose we use the π^0 decay experiment again. On the track that the electron and the positron travel, set electron detector A and positron detector B, in the direction of unit vector $\boldsymbol{a}$ and $\boldsymbol{b}$.

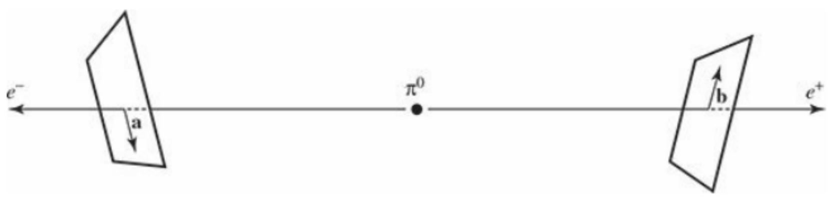

Figure 12.2: Bell's version of the EPRB experiment: detectors independently oriented in directions **a** and **b**.

(Griffiths, P569)

Each detector measures the spin of particle in its corresponding vector, and the result of measurement is either 1 or -1. The example in previous section is the special case of this thought experiment that $\boldsymbol{a} = \boldsymbol{b}$, in which after measuring, we always find one particle spin-up and the other spin-down. If we list result and calculate the product of results, we should have something like:

electron	positron	product (P)
1	-1	-1
1	1	1
-1	1	-1
-1	-1	1

In the case $\boldsymbol{a} = \boldsymbol{b}$, the product $P(\boldsymbol{a}, \boldsymbol{b})$ is always -1. On the other hand, if $\boldsymbol{a} = -\boldsymbol{b}$ (anti-parallel), then the product $P(\boldsymbol{a}, \boldsymbol{b})$ is always 1. Actually, $P(\boldsymbol{a}, \boldsymbol{b}) = -\boldsymbol{a} \cdot \boldsymbol{b}$. In 1964, J. S. Bell showed that this result is incompatible with any hidden variable theory.

The prove is as the following: suppose a hidden variable λ deterministically decides the measurement of the spin of two particles. Say, $A(\boldsymbol{a}, \lambda)$ and $B(\boldsymbol{b}, \lambda)$ which take value ± 1 are the outputs of measurements on electron and positron. When the detectors are aligned, the results are perfectly anti-correlated:

$$A(\boldsymbol{a}, \lambda) = -B(\boldsymbol{a}, \lambda) \rightarrow |A(\boldsymbol{a}, \lambda) A(\boldsymbol{b}, \lambda)| = 1$$

Regardless of the value of λ. And by the hidden variable theory, the average of the product of the measurements is

$$P(\boldsymbol{a}, \boldsymbol{b}) = \int \rho(\lambda)\, A(\boldsymbol{a}, \lambda) B(\boldsymbol{b}, \lambda) d\lambda$$

Where $\rho(\lambda)$ is the probability density for the hidden variable. Thus

$$P(\boldsymbol{a}, \boldsymbol{b}) = -\int \rho(\lambda)\, A(\boldsymbol{a}, \lambda) A(\boldsymbol{b}, \lambda) d\lambda$$

And suppose $\boldsymbol{c}$ is some other unit vector, then

$$P(\boldsymbol{a}, \boldsymbol{b}) - P(\boldsymbol{a}, \boldsymbol{c}) = -\int \rho(\lambda)\, [A(\boldsymbol{a}, \lambda) A(\boldsymbol{b}, \lambda) - A(\boldsymbol{a}, \lambda) A(\boldsymbol{c}, \lambda)] d\lambda$$

Since $A(\boldsymbol{b}, \lambda) = \pm 1, \left(A(\boldsymbol{b}, \lambda)\right)^2 = 1$, we may write

$$P(\boldsymbol{a}, \boldsymbol{b}) - P(\boldsymbol{a}, \boldsymbol{c}) = -\int \rho(\lambda)\, [A(\boldsymbol{a}, \lambda) A(\boldsymbol{b}, \lambda) - A(\boldsymbol{a}, \lambda) A(\boldsymbol{c}, \lambda)] d\lambda$$

$$= -\int \rho(\lambda)\, \left[A(\boldsymbol{a}, \lambda)\left(A(\boldsymbol{b}, \lambda) - A(\boldsymbol{c}, \lambda)\right)\right] d\lambda$$

$$= -\int \rho(\lambda)\ \left[\frac{A(\boldsymbol{a},\lambda)}{A(\boldsymbol{b},\lambda)}\left(\left(A(\boldsymbol{b},\lambda)\right)^2 - A(\boldsymbol{b},\lambda)A(\boldsymbol{c},\lambda)\right)\right] d\lambda$$

$$= -\int \rho(\lambda)\ \left[A(\boldsymbol{a},\lambda)A(\boldsymbol{b},\lambda)\left(1 - A(\boldsymbol{b},\lambda)A(\boldsymbol{c},\lambda)\right)\right] d\lambda$$

Also because $\left(1 - A(\boldsymbol{b},\lambda)A(\boldsymbol{c},\lambda)\right) \geq 0$,

$$|P(\boldsymbol{a},\boldsymbol{b}) - P(\boldsymbol{a},\boldsymbol{c})| \leq \int \rho(\lambda)\ \left(1 - A(\boldsymbol{b},\lambda)A(\boldsymbol{c},\lambda)\right) d\lambda$$

Which is equivalent to, the Bell inequality:

$$|P(\boldsymbol{a},\boldsymbol{b}) - P(\boldsymbol{a},\boldsymbol{c})| \leq 1 + P(\boldsymbol{b},\boldsymbol{c})$$

Even we don't know anything about the hidden variable λ, we know that it should suffice this inequality in nature. But the quantum mechanical prediction is certainly incompatible with the inequality: suppose unit vector $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$ takes the following configure:

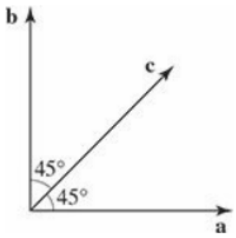

Figure 12.3: An orientation of the detectors that demonstrates quantum violations of Bell's inequality.

(Griffith, P571)

Then $P(a,b) = 0, P(a,c) = P(b,c) = -\frac{1}{\sqrt{2}} \approx -0.707$,

While the Bell's inequality states

$$0.707 \leq 1 - 0.707 = 0.293$$

A contradiction occurs. The quantum mechanics is thus incompatible with any hidden variable theory.

5. Conclusion

We examined the properties of two qubits entangled states and showed that by switching basis, we can adopt the entangled states as new basis to avoid the wavefunction collapse issue. We reviewed the EPR paradox and how Einstein wish to develop the hidden variable theories to complete the quantum physics and remove the indeterminacy. We lastly reviewed how Bell's inequality shattered Einstein's dream of physics theory unification, proving the incompatibility between quantum physics and hidden variable theories.

Whether Einstein wanted it or not, experiments forced us to accept the weird instant correlations among the measurement outcomes of entangled pairs. What if the world is inherently probabilistic? Not a big deal. The crucial step is to switch to the new perspective and accommodate it, like how Alice and Bob switch to the new basis, upon which life will certainly gain new meaning.

Reference:

[1] Preskill (2001). Lecture Notes for Ph219/CS219: Quantum Information and Computation Chapter 4 / John Preskill. California Institute of Technology

[2] Griffiths, & Schroeter, D. F. (2018). Introduction to quantum mechanics / David J. Griffiths and Darrell F. Schroeter. (Third edition.). Cambridge University Press.