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The Schwartzchild Geometry

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1 INTRODUCTION

In our study of general relativity, we are interested in the geometry of curved spacetime in certain special cases such as in the space around a spherical source of curvature. Einstein's equations in this case were first solved by Karl Schwartzchild in 1916. The resulting solution is known as the *Schwartzchild geometry*, and it is applicable in such cases as the space around a spherical star, or around a spherically symmetric black hole. Several significant conclusions can be drawn from this geometry, such as gravitational redshift of a light ray, or the precession of a planet's perihelion.

2 THE GEOMETRY

The line element of the Schwartzchild geometry is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + (\sin(\theta))^2 d\phi\right) \tag{2.1}$$

The above equation is of course expressed without the assumption of c = G = 1 units. The coordinates (t, r, θ, ϕ) are the *Schwartzchild coordinates*, and will be always listed in the above order from here on. The metric $g_{\alpha\beta}$ associated with the geometry is the Schwartzchild metric.

2.1 PROPERTIES OF THE METRIC

1. *Time Independence*. The metric has no dependence on the time coordinate *t*. The Killing vector associated with this symmetry is $\xi^\alpha = (1, 0, 0, 0)$.

2. *Spherical Symmetry.* If we hold t and r constant, the dt^2 and dr^2 terms in the above go to 0, and the resulting equation becomes:

$$d\Sigma^2 = r^2\left(d\theta^2 + (\sin(\theta))^2 d\phi\right) \tag{2.2}$$

Which is the typical geometry of an ordinary sphere of radius r in 3-D Euclidean space. As such the Schwartzchild metric is invariant under transformations in θ and ϕ. We can also see this in the ϕ case simply from the fact Equation 2.1 has no ϕ dependence. The ϕ symmetry has Killing vector $\eta^\alpha = (0,0,0,1)$.

3. *Classical Limit.* If we assume that GM/c^2r is small (i.e. M is small or r is large), then we can make use of the geometric series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad (\text{ when } x \in (0,1)) \tag{2.3}$$

We use the first two terms in place of $(1 - GM/c^2r)^{-1}$ in Equation 2.1 to obtain

$$ds^2 \approx -\left(1 - \frac{2GM}{c^2 r}\right)(c\,dt)^2 + \left(1 + \frac{2GM}{c^2 r}\right)dr^2 + r^2\left(d\theta^2 + (\sin(\theta))^2 d\phi\right) \tag{2.4}$$

Which is the form of the static weak field from a Newtonian potential $\Phi = -GM/r$. Because of this the constant M is interpreted as being the total mass of the source of curvature. In this case we define "mass" as the ability to produce curvature, which may include both mass in the typical sense as well as EM or nuclear energy. Furthermore the geometry is dependent only on total mass of the source, not on its distribution, a fact which is essentially a relativistic reformulation of Newton's shell theorem.

4. *The Schwartzchild Radius.* Note that as r approaches $2GM/c^2$, the dr^2 term in Eq. 2.1 approaches 0^{-1}. As such we expect there must be some sort of exceptional behavior at this radius. In the case of a spherical star, the radius turns out to not be terribly important; $r = 2GM/c^2$ is always inside the star itself, where a different metric applies and the Schwartzchild radius is no longer significant. In the case of a body collapsing into a black hole it is considerably more important, but here we only deal with the case of a star.

Lastly for this section we will rewrite Eq. 9.1 using units where c = G = 1, and give an explicit form for the metric $g_{\alpha\beta}$:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)(dt)^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + (\sin(\theta))^2 d\phi\right) \tag{2.5}$$

$$g_{\alpha\beta} = \begin{bmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2(\sin(\theta))^2 \end{bmatrix} \tag{2.6}$$

From here we explore various consequences of this geometry.

3 Gravitational Redshift

To an observer at a far distance from a source of Schwartzchild curvature, a light ray emitted from a source close to the center of curvature will appear redshifted, i.e. its wavelength will appear to be longer than the wavelength seen from the source. This effect, the gravitational redshift, can be approximated with the equivalence principle, but can also be precisely calculated via the Schwartzchild geometry. The frequency ω of the photon can be directly related to its energy, and as such the redshift can be quantified via the effect of gravity on the photon's energy. If an observer has 4-velocity $\mathbf{u}_{obs}$ and 4-momentum $\mathbf{p}$, then its energy can be given as:

$$E = -\mathbf{p}\cdot\mathbf{u}_{obs} \tag{3.1}$$

This can of course be related to frequency by the familiar relation $E = \hbar\omega$. If we assume the observer is stationary, then the spatial coordinates of $\mathbf{u}_{obs}$ are all zero, and the time component can be determined from the normalization condition:

$$\mathbf{u}_{obs}(r)\cdot\mathbf{u}_{obs}(r) = g_{\alpha\beta}u^{\alpha}_{obs}(r)u^{\beta}_{obs}(r) = -1 \tag{3.2}$$

With the spatial components all zero, this gives:

$$g_{tt}(r)\left[u^{t}_{obs}(r)\right]^2 = -1 \tag{3.3}$$

Which, substituting in the value of g_{tt} gives:

$$u^{t}_{obs}(r) = \left(1 - \frac{2M}{r}\right)^{-1/2} \tag{3.4}$$

Meaning the full velocity vector is:

$$\mathbf{u}_{obs}(r) = \left(1 - \frac{2M}{r}\right)^{-1/2}\boldsymbol{\xi} \tag{3.5}$$

Where $\boldsymbol{\xi}$ is the Killing vector for the time symmetry. So for a stationary observer at a radius r, we have

$$\hbar\omega = \left(1 - \frac{2M}{r}\right)^{-1/2} - (\boldsymbol{\xi}\cdot\mathbf{p}) \tag{3.6}$$

As $r \to \infty$, $(1-2M/r)^{-1/2} \to 1$ so the photon energy as seen by a distant observer is:

$$\hbar\omega_{\infty} = -(\boldsymbol{\xi}\cdot\mathbf{p}) \tag{3.7}$$

Note however that ($\boldsymbol{\xi}\cdot\mathbf{p}$ is conserved because of the symmetry in t, so it is the same everywhere. So if ω_* is the frequency seen by the emitter, the redshift is quantified as:

$$\omega_{\infty} = \omega_*\sqrt{1 - \frac{2M}{r}} \tag{3.8}$$

4 Precession of the Perihelion

The Schwartzchild geometry is useful in calculating the paths of massive particles moving within it, in other words timelike geodesics. Due to the geometries independence on t and ϕ, we have associated conserved quantities. These will be defined as:

$$e = -\boldsymbol{\xi} \cdot \mathbf{u} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} \tag{4.1}$$

$$l = \boldsymbol{\eta} \cdot \mathbf{u} = r^2(\sin(\theta))^2 \frac{d\phi}{d\tau} \tag{4.2}$$

These are associated with energy and angular momentum, respectively.

4.1 Effective Potential

As in Newtonian mechanics, we expect that all particle orbits will fall plane, so we will assume that this plane is $\phi = 0$, $\theta = \pi/2$ from here on out.

Another piece of information we can make use of in solving for these geodesics is the normalization of the 4-velocity:

$$\mathbf{u} \cdot \mathbf{u} = g_{\alpha\beta} u^\alpha u^\beta = -1 \tag{4.3}$$

Applying this, and the fact $\theta = \pi/2$ and $u^\theta = 0$ to the Schwartzchild metric yields:

$$-\left(1 - \frac{2M}{r}\right)\left(u^t\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(u^r\right)^2 + r^2\left(u^\phi\right)^2 = -1 \tag{4.4}$$

Making use of the identities given in Eq. 4.1 and Eq. 4.2, we can equivalently rewrite this as:

$$-\left(1 - \frac{2M}{r}\right)e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{l^2}{r^2} = -1 \tag{4.5}$$

Or equivalently,

$$\frac{e^2 - 1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right] \tag{4.6}$$

We make the following definitions:

$$\mathcal{E} \equiv \frac{e^2 - 1}{2} \tag{4.7}$$

$$V_{\text{eff}}(r) \equiv \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right] = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} \tag{4.8}$$

So we can write Eq. 4.6 as:

$$\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}(r) \tag{4.9}$$

Note that the expression used here for V_{eff} differs from the Newtonian equivalent only by the $-Ml^2/r^3$ term, making it only significant for large M or small r. We can solve for the extema of V_{eff} by solving $dV_{\text{eff}}/dr = 0$:

$$r_{\text{max}}^{\text{min}} = \frac{l^2}{2M}\left[1 \pm \sqrt{1 - 12\left(\frac{M}{l}\right)^2}\right] \tag{4.10}$$

If $l/M < \sqrt{12}$, then both of these values are complex and V_{eff} is negative everywhere. If $l/M > \sqrt{12}$, the maximum and minimum exist and are both real.

As in Newtonian mechanics, the behavior of the orbit depends on the relationship between $\mathscr{E}$ and V_{eff}. Turning points occur when $\mathscr{E} = V_{\text{eff}}$. If $l/M < \sqrt{12}$, this will not occur as long as $\mathscr{E}$ is positive, so an inwardly directed particle will fall directly into the source of curvature, in contrast to Newtonian predictions. When the effective potential is at an extremum, circular orbits are possible, with the maximum orbit being unstable and the minimum stable.

4.2 Radial Plunge Orbits

Consider a particle with $l = \mathscr{E} = 0$, falling from ∞ to the center of gravitational attraction. We then have that:

$$0 = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 - \frac{M}{r} \tag{4.11}$$

So the 4-velocity of the particle is:

$$u^\alpha = ((1-2M/r)^{-1}, -(2M/r)^{1/2}, 0, 0) \tag{4.12}$$

By integrating Eq. 4.11, we can obtain an expression for r as a function of τ:

$$(3/2)^{2/3}(2M)^{1/3}(\tau_* - \tau)^{2/3} \tag{4.13}$$

Where T_* is an integration constant. We can calculate dt/dr from the definition of e given in Eq. 4.1:

$$\frac{dt}{dr} = -\left(\frac{2M}{r}\right)^{-1/2}\left(1 - \frac{2M}{r}\right)^{-1} \tag{4.14}$$

Which we can integrate again to obtain:

$$t = t_* + 2M\left[-\frac{2}{3}\left(\frac{r}{2M}\right)^{3/2} - 2\left(\frac{r}{2M}\right)^{1/2} + \ln\left|\frac{\sqrt{r/2M}+1}{\sqrt{r/2M}-1}\right|\right] \tag{4.15}$$

Where t_* is a second constant of integration. We note a few interesting facts. Starting from any fixed value of r, it will take only finite proper time to reach the Schwartzchild radius of $2M$. The same movement, however, will take infinite coordinate time. This indicates a flaw in the Schwartzchild coordinates near the pivotal radius of 2M.

4.3 Stable Circular Orbits

As discussed previously, a stable circular orbit exists at the minimum radius r_{min} where the effective potential has its minimum value. As l/M decreases these radii become smaller, but there is a specific minimum for $l/M = \sqrt{12}$:

$$r_{\text{ISCO}} = 6M \tag{4.16}$$

We define angular velocity Ω:

$$\Omega \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\left(\frac{l}{e}\right) \tag{4.17}$$

We know that effective potential must be at a minimum, so we have from Eq. 4.10 a relationship for r and l. Secondly we know $\mathscr{E} = V_{\text{eff}}(r_{min})$, giving us:

$$e^2 = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right) \tag{4.18}$$

Meaning that for a circular orbit we have:

$$\frac{l}{e} = \sqrt{Mr}\left(1 - \frac{2M}{r}\right)^{-1} \tag{4.19}$$

Which implies the relation:

$$\Omega^2 = \frac{M}{r^3} \tag{4.20}$$

Which is another form for Kepler's third law. The 4-velocity of a particle in a circular orbit is given by:

$$u^\alpha = \left(\sqrt{1 - (3M/r)}\right)(1, 0, 0, \Omega) \tag{4.21}$$

4.4 Bound Orbits

To determine the shape of an orbit we must find either a function $\phi(r)$ or $r(\phi)$. We can do this from the relations given in Eq. 4.2 and Eq. 4.9 to obtain:

$$\frac{d\phi}{dr} = \pm\frac{l}{r^2}\frac{1}{\sqrt{2(\mathscr{E} - V_{\text{eff}}(r))}} = \pm\frac{l}{r^2}\left[e^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{l^2}{r^2}\right)\right]^{-1/2} \tag{4.22}$$

Which can be integrated to obtain $\phi(r)$ in the form of an elliptical integral. We will say that an orbit closes if the change in angle between successive inner or outer turning points is 2π. If it does not close, then the angle of precession is the difference between $\Delta\phi$ and 2π:

$$\delta\phi_{\text{prec}} = \Delta\phi - 2\pi \tag{4.23}$$

If r_1 is an inner turning point and r_2 is an outer turning point, then:

$$
\begin{aligned}
\Delta\phi &= 2l\int_{r_1}^{r_2} dr\,\frac{1}{r^2}\left[e^2-\left(1\frac{2M}{r}\right)\left(1+\frac{l^2}{r^2}\right)\right]^{-1/2} \\
&= 2l\int_{r_1}^{r_2} dr\,\frac{1}{r^2}\left[c^2\left(e^2-1\right)+\frac{2GM}{r}-\frac{l^2}{r^2}+\frac{2GMl^2}{c^2r^3}\right]^{-1/2}
\end{aligned}
\tag{4.24}
$$

The second form of the above is obtained simply by reintroducing c and G explicitly. By expanding this integral we obtain an expression for $\delta\phi_{\text{prec}}$:

$$
6\pi\left(\frac{GM}{cl}\right)^2 = \frac{6\pi G}{c^2}\frac{M}{a(1-\epsilon^2)} \tag{4.25}
$$

Where a is the semimajor axis and ϵ is the eccentricity of the orbit. The relativistic effect is proportional to $1/r^3$, so in the Solar system it is only significant enough to be observable in the case of Mercury.

5 Light Orbits

The orbits of a light ray under the Schwartzchild geometry share some significant similarities to the orbits of a particle, although there are key differences as well. As before we have conserved quantities associated with the symmetries in ϕ and t. The conserved quantities are:

$$
e \equiv -\boldsymbol{\xi}\cdot\mathbf{u} = \left(1-\frac{2M}{r}\right)\frac{dt}{d\lambda} \tag{5.1}
$$

$$
l \equiv \boldsymbol{\eta}\cdot\mathbf{u} = r^2(\sin(\theta))^2\frac{d\phi}{d\lambda} \tag{5.2}
$$

Where λ is an affine parameter. The last requirement is also similar to one used in the particle case:

$$
\mathbf{u}\cdot\mathbf{u} = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0 \tag{5.3}
$$

The derivation followed is also similar. We start from Eq. 5.3 and apply the Schwartzchild metric and the condition that $\theta = \pi/2$. We then substitute in for e and l.

$$
-\left(1-\frac{2M}{r}\right)^{-1} e^2\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2+\frac{l^2}{r^2} = 0 \tag{5.4}
$$

We define quantities b and $W_{\text{eff}}(r)$:

$$
b \equiv \left|\frac{l}{r}\right| \tag{5.5}
$$

$$
W_{\text{eff}}(r) \equiv \frac{1}{r^2}\left(1-\frac{2M}{r}\right) \tag{5.6}
$$

W_{eff} here corresponds to the effective potential, and b corresponds to the impact parameter of the light ray. From this we rewrite the above equation as:

$$\frac{1}{b^2} = \frac{1}{l^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{\text{eff}}(r) \tag{5.7}$$

$W_{\text{eff}}(r)$ has a maximum at $r = 3M$:

$$W_{\text{eff}}(3M) = \frac{1}{27M^2} \tag{5.8}$$

It is possible for light to have a circular orbit in the case of $b^2 = 27M^2$, but these orbits are unstable and cannot occur around a sun-like star (though it can around a black hole). If $1/b^2 < 1/27M^2$ then the light ray will have a turning point and then escape the gravity of the curvature source. If $1/b^2 > 1/27M^2$, then a plunging orbit will be observed.

5.1 Deflection of Light

When a light ray passes close to a source of curvature, it gets deflected by it. We quantify this in terms of an angle $\delta\phi_{\text{def}}$, defined to be the angle between the initial and final trajectories of the light ray. As with particle orbits, we do this by solving for $d\phi/dr$ from equations seen earlier:

$$\frac{d\phi}{dr} = \pm\frac{1}{r^2}\left[\frac{1}{b^2} - W_{\text{eff}}(r)\right]^{-1/2} \tag{5.9}$$

The light ray comes in from infinity, passes through a turning point r_1, and goes out to infinity again. So the change in ϕ is:

$$\Delta\phi = 2\int_{r_1}^{\infty} dr \frac{1}{r^2}\left[\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)\right]^{-1/2} \tag{5.10}$$

For small values of M/b, this integral can be approximately evaluated as:

$$\Delta\phi \approx \pi + \frac{4M}{b} \tag{5.11}$$

So $\delta\phi_{\text{def}} \approx 4M/b$. Reinseting G and c explicitly gives us:

$$\delta\phi_{\text{def}} \approx \frac{4GM}{c^2 b} \tag{5.12}$$

Experimental observation of this effect provided a significant piece of evidence that relativity was true.

5.2 Delay of Light

Because of the same physical effect which causes the deflection of light around sources of curvature, light can also be delayed in passing from one point to another and back if its path takes it close to a large mass. Consider a pulse of light sent from Earth to a satellite orbiting on the far side of the sun. The satellite reflects the signal back to Earth where it is detected again. Let r_E and r_S denote the orbital radii of the Earth and the satellite, respectively, and let r_0 denote the radius of closest approach of the light ray to the sun.

The round-trip travel time is not long compared to the length of a year, so we can safely make the assumption that the Earth remains stationary in the time needed for the light ray to travel out and return. Similar to previous calculations, we find the time difference between the ray's departure and return by finding t as a function of r. From equations 5.1 and 5.7:

$$\frac{dt}{dr} = \pm\frac{1}{b}\left(1-\frac{2M}{r}\right)^{-1}\left[\frac{1}{b^2} - W_{\text{eff}}(r)\right]^{-1/2} \tag{5.13}$$

Let $t(r_2, r_1)$ denote the time needed for the ray to pass from a radius r_1 to a radius r_2. Total time elapsed is then:

$$\Delta t = 2t(r_e, r_0) + 2t(r_0, r_S) \tag{5.14}$$

The equation for $t(r_2, r_1)$

$$t(r_2, r_1) = \int_{r_1}^{r_2} dr \frac{1}{b}\left(1-\frac{2M}{r}\right)^{-1}\left[\frac{1}{b^2} - W_{\text{eff}}(r)\right]^{-1/2} \tag{5.15}$$

And where $1/b^2 = W_{\text{eff}}(r_1)$. Approximating this integral in a fashion similar to what was done in previous sections, we obtain:

$$t(r_2, r_1) = \sqrt{r_2^2 - r_1^2} + 2M\ln\left[\frac{r_2 + \sqrt{r_2^2 - r_1^2}}{r_1}\right] + M\sqrt{\frac{r_2 - r_1}{r_2 + r_1}} \tag{5.16}$$

The first term is what we expect from the non-relativistic case of a ray moving in a straight line through normal Euclidean space at c. The remaining terms provide relativistic corrections. We express the effect in terms of the difference of classical and relativistic predictions:

$$(\Delta t)_{\text{excess}} = \Delta t - 2\sqrt{r_E^2 - r_0^2} - 2\sqrt{r_S^2 - r_0^2} \tag{5.17}$$

If we assume that the ray moves quite close to the sun, i.e. r_0 is small, we get the approximation:

$$(\Delta t)_{\text{excess}} \approx \frac{4GM}{c^3}\left[\ln\left(\frac{4r_S r_E}{r_0^2}\right) + 1\right] \tag{5.18}$$

With c and G written explicitly.

6 Sources

1. Hartle, J. B. (2003). *Gravity: An Introduction to Einstein's General Relativity*, San Francisco: Addison Wesley.