Free Field Theory and Propagators

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Abstract

We take the continuum limit of the mattress path integral model and show that it reduces to the classical field equation in the limit $\hbar \to 0$. We then introduce the source function $J(x)$ as a means to create and annihilate particles. We solve our path integral under the free field condition and get the Klein-Gordon Equation. We then define the propagator and evaluate it in free field theory using the method of contours.

1 Introduction

This paper is adapted from Anthony Zee *Quantum Field Theory in a Nutshell* chapter I.3. The continuum limit is the basis for QFT. Once we have the path integral, we are ready to start making physical predictions. The path integral we get is only solvable for free field theory which can only be used to describe a single relativistic, massive particle. Studying the behavior and methodology of free field theory is useful for understanding for complex theories which describe scattering processes.

2 The Continuum Limit

In our mattress model we derived the path integral for a single particle

$$\mathrm{Z} = \int Dq(t) \exp[i \int_0^T dt(\tfrac{1}{2}m\,(dq/dt)^2 - V(q))] \qquad (1)$$

We can easily generalize this to N particles with the new Hamiltonian

$$H = \sum_a \frac{1}{2m_a} p_a{}^2 + V(q_1, q_2, ..., q_N) \qquad (2)$$

We use a to label the particle's positions and momenta. Substituting back into the integral, we get

$$Z = \int Dq(t)\, \mathrm{ex}[(i \int_0^T dt(\sum_a \tfrac{1}{2} m_a (\tfrac{dq_a}{dt})^2 - V(q_1, q_2, ..., q_N))] \qquad (3)$$

Which we simplify by defining the action

$$S(q) = \int_0^T dt(\sum_a \tfrac{1}{2} m_a (\tfrac{dq_a}{dt})^2 - V(q_1, q_2, ..., q_N)) \qquad (4)$$

Thus we have

$$Z = \int Dq(t) e^{iS(q)} \qquad (5)$$

Note that the potential energy now includes interaction energy terms between particles which take the form $v(q_a - q_b)$ as well as the eternal potential energy terms which take the form $w(q_a)$. We take the special case where V has the form

$$V(q_1, q_2, ..., q_N) = \sum_{ab} \tfrac{1}{2} k_{ab}(q_a - q_b)^2 + ... \qquad (6)$$

Which is the generalized mattress potential. We are almost at QFT! If we consider phenomena on scales much larger than that of the lattice spacing, we take the limit $l \to 0$. The label a goes to the vector $\boldsymbol{x}$[1]. By convention we replace q with φ, so we have $q_a(t) \to \varphi(t, \boldsymbol{x})$, a field.

The kinetic energy goes like $\Sigma_a \frac{1}{2} m_a (\frac{dq}{dt})^2 \to \int d^2x \frac{1}{2} \sigma(\partial\varphi/\partial t)^2$. Where we replace a sum with an integral and write mass per unit area as $\sigma = m_a/l^2$. For simplicity we assume all the masses are equal, otherwise sigma would be a function of $\boldsymbol{x}$ and we would have an inhomogeneous system which would make writing down the invariant action quite difficult.

Looking at the first term in V, we assume that k_{ab} only connects the nearest lattice points. Thus for nearest neighbor points we have $(q_a - q_b)^2 \simeq l^2(\partial\varphi/\partial x)^2 + \dots$. Where the derivatives are taken in the direction that joins a and b.

With this limit, we write the action

$$S(q) \to S() \equiv \int_0^T dt \int d^2x \mathcal{L}()$$

$$= \int_0^T dt \int d^2x \tfrac{1}{2}\{(\tfrac{\partial}{\partial t})^2 - [(\tfrac{\partial}{\partial x})^2 + (\tfrac{\partial}{\partial y})^2] - {}^2 - {}^4 + ...\} \quad (7)$$

Where ϱ is determined by some uninteresting relation between k_{ab} and l. We let $T \to \infty$ so that we integrate over all of spacetime. We simplify by writing $\varrho = \sigma c^2$ and scale $\varphi \to \varphi/\sqrt{}$. Thus the term $(\partial/\partial t)^2 - c^2[(\partial/\partial x)^2 + (\partial/\partial y)^2]$ appears in the lagrangian. We find that c has dimensions of velocity and determines the phase velocity of the waves in the mattress.

The mattress is nonphysical. We just used it for pedagogical reasons. In the modern view (Landau-Ginsburg), we start with a symmetry and choose our fields by defining how they transform under the symmetry. We then write down the action using, at most, second order time derivatives. In our case we chose Lorentz invariance as our symmetry and scalar fields as our fields. We get the action ($c = 1$)

$$S = \int d^dx [\tfrac{1}{2}(\partial)^2 - \tfrac{1}{2}m^2{}^2 - \tfrac{g}{3!}{}^3 - \tfrac{\lambda}{4!}{}^4 + ...] \qquad (8)$$

With various numerical factors that we will use later. Here we are working in $d = D + 1$ dimensional spacetime. Usually we consider $d = 4$.

The symmetry restrictions are very powerful. We find that the lagrangian must have the form

$$\mathcal{L} = \tfrac{1}{2}(\partial)^2 - V() \qquad (9)$$

We restrict V to be a polynomial of φ for simplicity but it will not affect most of our discussion.

[1] In this paper I will use plainface for 4-vectors and bold for 3-vectors

We now see why the variable q rather than $\boldsymbol{x}$ was used to denote position. In QFT $\boldsymbol{x}$ is a label rather than a dynamical variable, here the field φ is the dynamical variable.

We can summarize the continuum limit in the table

$q \rightarrow \varphi$
$a \rightarrow \boldsymbol{x}$
$q_a(t) \rightarrow \varphi(t,\boldsymbol{x}) = \varphi(x)$
$\sum_a \rightarrow \int d^D x$

(10)

Thus we get the continuum path integral for d dimensional spacetime

$$Z = \int D \exp[i\int d^d x(\tfrac{1}{2}(\partial)^2 - V()\,] \qquad (11)$$

Note that we get quantum mechanics for $d = 1$.

3 The Classical Limit

We can take the classical limit of the path integral formalism as a sanity check. For convenience we have used units where $\hbar = 1$, we now return to SI units and put $\hbar$ back into the path integral

$$Z = \int D \exp\,(i/\hbar)\int d^4 x \mathcal{L}() \qquad (12)$$

We then take the limit $\hbar << S$. The integral can be evaluated using the stationary phase approximation. Using the Euler-Largrange variational procedure, we get

$$\partial_\mu \frac{\mathcal{L}}{(\partial_\mu)} - \frac{\mathcal{L}}{} = 0 \qquad (13)$$

The classical field equation, as expected. In our scalar field theory this is

$$(\partial^2 + m^2)(x) + \tfrac{g}{2}(x)^2 + \tfrac{\lambda}{6}(x)^3 + ... = 0 \qquad (14)$$

4 Creation and Annihilation

We now want to do some physics. Lets create a particle at some point in spacetime, watch it move, and then annihilate it at some later point. In our pedagogical mattress, we create some excitations by pushing some mass labeled by a. This adds a term $\Sigma_a J_a(t)\, q_a$ to the potential $V(q)$. taking the continuum limit, we add the term $\int d^D x J(x)(x)$ to the lagrangian. We call $J(t,x)$ a source function, and it describes how the mattress is pushed. We get the path integral

$$Z = \int D \exp\,[i\int d^4 x[\tfrac{1}{2}(\partial)^2 - V() + J(x)(x)]\,] \qquad (15)$$

5 Free Field Theory

The above integral is only solvable when

$$\mathcal{L}() = \tfrac{1}{2}[(\partial)^2 - m^{2\,2}] \quad (16)$$

We call this theory free or Gaussian theory. The field equation comes out to the Klein-Gordon equation $(\partial^2 + m^2) = 0$. Since it is linear we immediately find the solution $(, t) = e^{i(t - \cdot)}$, where (for $\hbar = 1$)

$$^2 = {}^2 + m^2 \quad (17)$$

In the natural system of units, frequency ω is the energy $\hbar\omega$, and wave vector $\boldsymbol{k}$ equals momentum $\hbar\boldsymbol{k}$. Thus we have just found the energy-momentum relation. This theory should describe a relativistic particle of mass m.

We now do the path integral

$$Z = \int D \exp\left[i \int d^4x \{ \tfrac{1}{2}[(\partial)^2 - m^{2\,2}] + J \}\right. \qquad (18)$$

Integrate by parts under $\int d^4x$ and assume boundary terms are zero, we get

$$Z = \int D \exp\{ i \int d^4x [-\tfrac{1}{2}(\partial^2 + m^2) + J] \} \qquad (19)$$

To solve this integral, we use a discretization trick. Let $\varphi(x) \to \varphi_i = \varphi(ia)$ for i an integer and a the lattice spacing. Differential operators become matrices, for example $\partial(ia) \to (1/a)({}_{i+1} - {}_i) \equiv \Sigma_j M_{ij\,j}$, for some matrix M. Unsurprisingly, integrals go to sums, for example $\int d^4x J(x)(x) \to a^4 \Sigma_i J_{i\,i}$.

The path integral is just an integral we did earlier

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} dq_1 dq_2 \ldots dq_N e^{(i/2) q \cdot A \cdot q + iJ \cdot q}$$

$$= \left(\frac{(2i)^N}{det(A)}\right)^{\frac{1}{2}} e^{-(i/2) J \cdot A^{-1} \cdot J} \qquad (20)$$

With $A \to -(\partial^2 + m^2)$, and the inverse equation $A_{ij} A_{ik}^{-1} = {}_{ik}$ going to

$$-(\partial^2 + m^2) D(x - y) = {}^{(4)}(x - y)$$

in the continuum limit. The continuum limit of A_{jk}^{-1} is denoted by the function $D(x - y)$ which we call the propagator. We will return to the propagator momentarily. We note that the propagator is a function of the displacement of x and y and not just x since no point in spacetime is special. Also note that Kronecker deltas are replaced by Dirac deltas.

The final result is

$$Z(J) = \mathsf{C} \exp\ [-(i/2) \iint d^4x d^4y J(x) D(x - y) J(y) \equiv \mathsf{C} e^{iW(J)} \qquad (21)$$

The factor C is uninteresting and will be omitted from now on. $\mathsf{C} = Z(J = 0)$ and we define $W(J)$ such that

$$Z(J) \equiv Z(J = 0) e^{iW(J)} \qquad (22)$$

Thus

$$W(J) = -\tfrac{1}{2} \iint d^4x d^4y J(x) D(x - y) J(y) \qquad (23)$$

Which is a quadratic functional of J. However, Z(J) depends on arbitrarily high powers of J. This will be of importance in the section on Feynman diagrams.

6 The Propagator

We return to the propagator $D(x-y)$. Since it is the inverse of a differential operator, it is closely related to Green's functions.

Our path integral is easier to solve in momentum space, so we Fourier transform and write the Dirac delta function in integral form

$$-(\partial^2 + m^2)D(x-y) = \int \frac{d^4k}{(2)^4} e^{ik(x-y)} = {}^{(4)}(x-y) \qquad (24)$$

The solution is

$$D(x-y) = \int \frac{d^4k}{(2)^4} \frac{e^{ik(x-y)}}{k^2-m^2}$$

But wait! We have to be careful here as the integral hits a pole. To avoid this we replace $m \rightarrow m - i\varepsilon$ with ε infinitesimally small. We also note reflection symmetry for momentum[2] $k \rightarrow -k$.

We first integrate over k^0 using the Cauchy integral formula. We first write k as $k = k^0 + \boldsymbol{k}$. Thus the denominator becomes $1/(k^2 - m^2 + i)$. Define ω_k $\equiv + \sqrt{^2 + m^2}$. We have poles at $\pm\sqrt{{}_k^2 - i}$. For the ε small limit these reduce to $+{}_k - i$ and $-{}_k + i$. For $\varepsilon > 0$ we have one pole in the upper half-plane and one in the lower half-plane. Integrating over real k^0 from $-\infty$ to ∞ we encounter no poles. We just need to close the integration contour.

For $x^0 > 0$, $\exp(ik^0x^0)$ is exponentially damped for k^0 in the upper half-plane, and vice versa for $x^0 < 0$. Thus we choose the contours to be semicircles with infinite radius in the upper half-plane pole and the lower half-plane pole respectively. From Cuachy's integral formula we get $-i\int \frac{d^3k}{(2)^3 2_k} e^{-i({}_k t - \cdot)}$ and $-i\int \frac{d^3k}{(2)^3 2_k} e^{+i({}_k t - \cdot)}$ respectively. Using the Heaviside step function we write our answer in one line

$$-i\int \frac{d^3k}{(2)^3 2_k}[e^{-i({}_k t - \cdot)}(x^0) + e^{+i({}_k t - \cdot)}(-x^0)$$

$D(x-y)$ describes the probability amplitude for a disturbance in the field to propagate from the y to x.

7 References

Zee, A. Quantum Field Theory in a Nutshell, 1st ed., Princeton University Press, 2010, pp. 17-24.

[2] Here it should be obvious that I mean *energy*-momentum.