Canonical Quantization & The Path Integral Formulation ; A Brief Comparison

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"I have an equation, do you have one too?"

— Dirac, allegedly to Feynman

1 Introduction

There have been, historically, two separate methods of arriving at quantum field theory, hence force referred to as QFT. The first method is known as canonical quantization, and was first used by Paul Dirac to derive quantum electrodynamics in 1927. This formulation takes a direct route to QFT by quantizing classical systems directly, with an effort to maintain maximal symmetries. The later method is known as the Path Integral Formulation, completed and popularized by Richard Feynman in 1948. In this formulation, QFT is derived by quantizing another classical concept: the principle of least action — not by quantizing the system directly. The following is based on both my second and my first lectures for Kapitza Fall 2020. I will briefly explain the beginning of both formulations.

2 Canonical Quantization

The first method used to derive the beginnings of QFT is canonical quantization. We can begin with a reminder of the operators in a classical system. Defining the Lagrangian and Hamiltonian:

$$\mathcal{L} = \frac{m}{2}\dot{q}^2 - V(q) \qquad\qquad \mathcal{H} = p\dot{q} - \mathcal{L} = \frac{p^2}{2m} + V(q)$$

Heisenberg's approach to quantum mechanics gives the following commutation relation. Recall the product rule for commutators:

$$[p, q] = -i\hbar \tag{1}$$

$$[AB, C] = A[B, C] + [A, C]B \qquad\qquad [X, YZ] = Y[X, Z] + [X, Y]Z$$

Already, with this first imposition, commutation relations with the Hamiltonian becomes an operators on a state. For example, taking the time derivative of either generalized position or generalized momentum is analogous to using the following commutation relation. (Note the extra factor of $\frac{i}{\hbar}$.)

$$\frac{i}{\hbar}[\mathcal{H}, p] = \frac{i}{\hbar}[\frac{p^2}{2m} + V(q), p] = \frac{i}{2m\hbar}[p^2, p] + \frac{i}{\hbar}[V(q), p] = \frac{i}{\hbar}[V(q), p] = -V'(q)$$

$$\frac{i}{\hbar}[\mathcal{H}, q] = \frac{i}{\hbar}[\frac{p^2}{2m} + V(q), q] = \frac{i}{2m\hbar}[p^2, q] = \frac{i}{2m\hbar}\left(p[p, q] + [p, q]p\right) = \frac{i}{2m\hbar}\left(-\frac{2i}{\hbar}p\right) = \frac{p}{m} = \dot{q}$$

Alternatively, one can think of operators composed of p and q as evolving according to some operator $O(t) = e^{i\mathcal{H}t}O(0)e^{-i\mathcal{H}t}$ Combining the above commutators, we arrive at the operator equation of motion: $\ddot{q} = -V'(q)$.

With hindsight, we can define a new operator, composed from p and q, called a, shown below in natural units with unit mass. Using the commutation relation (1), we arrive at the conclusion that a has a unit commutator with its adjoint.

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip) \tag{2}$$

$$[a, a^\dagger] = \frac{1}{2\omega}(\omega q + ip)(\omega q - ip) - \frac{1}{2\omega}(\omega q - ip)(\omega q + ip)$$

$$= \frac{1}{2\omega}\left(\omega^2 q^2 - p^2 - i\omega qp + i\omega pq - \omega^2 q^2 + p^2 - i\omega qp + i\omega pq\right) = \frac{i}{2\omega}\left(2\omega qp - 2\omega pq\right) = -i[p, q] = 1$$

Using the time evolution operator on a we arrive at the following:

$$\frac{d}{dt}a = i[\mathcal{H}, a] = -i\sqrt{\frac{\omega}{2}}\left(ip + \frac{1}{\omega}V'(q)\right) \tag{3}$$

Conveniently, the above relation can once again be written in terms of a and its adjoint $a^\dagger$. The above operator can most suitably be applied to a simple harmonic oscillator (SHO). Such, we define $V'(q) = \omega^2 q$ and the value of (3) simplifies down to $-i\omega a$. The Hamiltonian then becomes $\omega(a^\dagger a + \frac{1}{2})$.

Taking the same approach a step further, we can generalize to multiple dimensions. For a system with an arbitrary number of particles, the Lagrangian becomes the following.

$$\mathcal{L} = \sum_a \frac{1}{2}\dot{q}_a^2 - V(q_1, q_2, \ldots, q_N)$$

The commutation relation shown in (1) can then be generalized to multiple dimensions also.

$$[p_a(t), q_b(t)] = -i\delta_{ab} \tag{4}$$

To get to a field theory, generalized position, q, and momentum, p, are replaced with canonical position and momentum densities ϕ and π respectively.

$$\mathcal{L} = \int d^D x\{\frac{1}{2}(\dot{\phi}^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2) - u(\phi)\}$$

The analogous central commutation relation is thus generalized to the following:

$$[\pi(\vec{x}, t), \phi(\vec{x}', t)] = [\pi(\vec{x}, t), \partial_0\pi(\vec{x}', t)] = -\delta^D(\vec{x} - \vec{x}') \tag{5}$$

Applying the analogous Legendre transformation, we can also obtain an expression for the Hamiltonian.

$$\mathcal{H} = \int d^D x[\pi(\vec{x}, t)\partial_0\phi(\vec{x}, t) - \mathcal{L}] = \int d^D x\{\frac{1}{2}(\pi^2 - (\vec{\nabla}\phi)^2 - m^2\phi^2) - u(\phi)\}$$

The u term in both the field Lagrangian and Hamiltonian represents the anharmonic term in the field. In other words, the interaction term in QFT.

Similar continuations can be performed on other multidimensional concepts such as creation and annihilation operators in order to derive QFT. In essence, canonical quantization takes the classical concepts of generalized position and momentum, then elevates them to multi-dimensions and finally continues them into a field. The Lagrangian, Hamiltonian, and other operators follow the same path. This then forms the basis of QFT.

3 The Path Integral Formulation

The path integral formulation begins with a more conceptual interpretation of the double slit experiment. Consider a particle starting at point $\mathcal{S}$ and travelling to some specific point $\mathcal{O}$ on a detector. In the normal experiment, there is a screen in between the start and end location that contains two slits, their positions denoted as $\mathcal{A}_1$ and $\mathcal{A}_2$. In this case, the amplitude, A, measured at $\mathcal{O}$ can then be described as the superposition of both possible paths that the particle can take. These being $(\mathcal{S} \to \mathcal{A}_1 \to \mathcal{O})$ and $(\mathcal{S} \to \mathcal{A}_2 \to \mathcal{O})$.

The question then becomes: *What happens when more slits are added?* The amplitude measured at $\mathcal{O}$ will still be the sum of all possible paths:

$$A(\mathcal{O}) = \sum_i (\mathcal{S} \to \mathcal{A}_i \to \mathcal{O})$$

The question then becomes: *What happens when more screens are added?* The amplitude measured at $\mathcal{O}$ will still be the sum of all possible paths:

$$A(\mathcal{O}) = \sum_i \sum_j \dots \sum_n (\mathcal{S} \to \mathcal{A}_i \to \mathcal{B}_j \to \dots \to \mathcal{Z}_n \to \mathcal{O})$$

Where $\mathcal{A}$ through $\mathcal{Z}$ denote each screen and i through n index each slit.

At a certain point, empty space can be considered as an infinite amount of screens, each having an infinite amount of slits. This is the beginning of the path integral formulation.

With the goal of calculating the same amplitude A described above, we transition to generalized coordinates, and consider a particle travelling from $(q_0 \to q_f)$ in some total amount of time T. This means that the path of the particle, and thus the measured amplitude is governed by the unitary operator: $e^{-i\mathcal{H}T}$.

Going off of the infinite screen and slit interpretation, a particle will encounter a new time evolution operator at 'every' instant. Thus, we express the total time as a sum of small time steps: $T = N\delta t$ and then allow the number of steps to go to infinity. The state of a particle is described as the following:

$$\langle q_f|e^{-i\mathcal{H}\delta t}e^{-i\mathcal{H}\delta t}\dots|q_0\rangle \tag{6}$$

Consider the following truth:

$$\int dq|q\rangle\langle q| = 1$$

The above operator has unit value and can therefore be happily inserted into (6) where required. In this way, we obtain the following:

$$\left(\prod_{j=1}^{N-1}\int dq_i\right)\langle q_f|e^{-i\mathcal{H}\delta t}|q_{N-1}\rangle\langle q_{N-1}|e^{-i\mathcal{H}\delta t}|q_{N-2}\rangle\dots\langle q_2|e^{-i\mathcal{H}\delta t}|q_1\rangle\langle q_1|e^{-i\mathcal{H}\delta t}|q_0\rangle \tag{7}$$

For a free particle, the Hamiltonian is just $\frac{\hat{p}^2}{2m}$. Evaluating a single time step of the above equations, we arrive at the following:

$$\langle q_{j+1}|e^{-i\frac{\hat{p}^2}{2m}\delta t}|q_j\rangle \tag{8}$$

Recall another useful relation:

$$\int\frac{dp}{2\pi}|p\rangle\langle p| = 1$$

In the same fashion as previously shown, the equation above can be inserted into (7) where necessary. The operator status of p drops away if acting on the eigenstate.

$$\int\frac{dp}{2\pi}\langle q_{j+1}|e^{-i\frac{\hat{p}^2}{2m}\delta t}|p\rangle\langle p|q_j\rangle = \int\frac{dp}{2\pi}e^{-i\frac{p^2}{2m}\delta t}\langle q_{j+1}|p\rangle\langle p|q_j\rangle$$

Using the fact that $\langle q|p\rangle = e^{ipq}$ we get:

$$\int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\delta t} e^{ip(q_{j+1}-q_j)}$$

The above takes the form of a Gaussian integral: $\int e^{ax^2+Jx}dx$ The solution to such integrals are known, and such we get the following new expression:

$$\left(\frac{-im}{2\pi\delta t}\right)^{\frac{1}{2}} e^{\frac{i\delta tm}{2}\left[\frac{q_{j+1}-q_j}{\delta t}\right]^2} \tag{9}$$

This expression concerns only a single timestep. Plugging this back into (7) yields the full expression.

$$\left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}} \left(\prod_{j=1}^{N-1} \int dq_i\right) e^{\frac{i\delta tm}{2}\sum_{j=0}^{N-1}\left[\frac{q_{j+1}-q_j}{\delta t}\right]^2} \tag{10}$$

The expression luckily simplifies quite nicely when we take $\delta t \to 0$

$$\int Dq(t) e^{i\int_0^T dt \frac{m}{2}\dot{q}^2} \tag{11}$$

$$\int Dq(t) = \lim_{N\to\infty} \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}} \left(\prod_{k=1}^{N-1} \int dq_k\right) \tag{12}$$

In (11) we get a familiar $\frac{m}{2}\dot{q}^2$ which arose because we were considering a free particle. If the Lagrangian is defined as as a Legendre transform of the Hamiltonian, (11) produces the classical action. Thus we obtain:

$$\int Dq(t) e^{i\int_0^T dt \mathcal{L}(q,\dot{q})} \tag{13}$$

In essence, the path integral formulation considers the path travelled by a particle to be the superposition of all possible paths that particle can take. From here, the basis of QFT is formed.

4 Conclusions

Although personal preference may vary between the two formulations, both offer their own advantages and disadvantages to understanding quantum phenomena. While Feynman's completion of the path integral formulation was initially met with skepticism by the scientific community, including Dirac himself, it has grown in popularity into the dominant formulation used today. However, by each of their own unique uses, it is unlikely that either will ever lose all attention in the foreseeable future.