Boltzmann Equation for Photons

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Abstract

This paper is based on my lecture for the Kapitza Society. In it, we discuss the collision-less Boltzmann equation for photons, its zero- and first-order equations, and the collision terms that will arise due to considerations of Compton Scattering to produce the (first-order) Boltzmann equation for photons. Dodelson's *Modern cosmology* is used as a reference for much of the content.

1 Introduction

In order to describe the particles (in this case photons) in a region of spacetime, we determine a distribution function $f(x^i, p^i)$ that maps a given event x^i and 4-momentum p^i to the number of particles in that state. The Boltzmann equation states that the rate of change of f along the particle world-lines is equal to the rate of change of f due to collisions, denoted $C(x^i, p^i, f)$ [2]. For our discussion of the Boltzmann equation for photons, we only need to deal with how the photons affected by the metric and collisions with electrons (via Compton Scattering).

2 The Collision-less Boltzmann Equation for Photons

From work in previous lectures, we arrive at the collision-less Boltzmann equation for photons (1):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} - p\frac{\partial f}{\partial p}[H + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}]. \tag{1}$$

This was accomplished by taking a description of the change in the momentum of a photon as it moves freely through a Friedmann Robertson Walker (FRW) Universe (2):

$$\frac{1}{p}\frac{dp}{dt} = -H - \frac{\partial \Phi}{\partial t} - \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i} \tag{2}$$

and substituting it into a previously derived form of the Boltzmann equation (3):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial p}\frac{dp}{dt}. \tag{3}$$

This equation (1) accounts for most physical effects that we want to account for, including the redshift of photons due to the expansion of the universe, and will allow us to determine the equations governing anisotropies. The first two terms on the right hand side of the equation lead to the continuity and Euler equations from hydrodynamics, the third term accounts for the photon's loss of energy due to the universe's expansion, and the last two terms account for regions of our distribution function that are under-/over-dense (due to our perturbation).

We begin our analysis by expanding our distribution function about its zero-order Bose-Einstein value. To this, we will describe f as a function of $\vec{x}, p, \hat{p},$ and t. We will use the zero-order temperature T as a function of only time (as such, it will scale like a^{-1}) This is valid because in the smooth, zero-order universe, photons are distributed homogeneously and isotropically. The perturbation will be denoted Θ (or equivalently $\frac{\delta T}{T}$), a function of $\vec{x}$ and $\hat{p}$ as it will account for inhomogeneities and anisotropies. It will turn out that the change in momentum of a photon due to Compton scattering will be negligible, and as such, we will write $\Theta = \Theta(\vec{x}, \hat{p}, t)$. With all of these considerations, we write:

$$f(\vec{x}, p, \hat{p}, t) = [exp\{\frac{p}{T(t)[1 + \Theta(\vec{x}, \hat{p}, t)]}\} - 1]^{-1}. \tag{4}$$

Next, we acknowledge that Θ is small, and expand while keeping only terms up to first order) and noticing that the zero-order distribution function is actually the Bose-Einstein distribution with no chemical potential:

$$f^{(0)} = [exp\{\frac{p}{T}\} - 1]^{-1}, \tag{5}$$

we can write:

$$\begin{aligned} f \simeq \frac{1}{e^{p/T} - 1} + (\frac{\partial}{\partial T}[exp\{\frac{p}{T}\} - 1]^{-1})T\Theta \\ = f^{(0)} + T\frac{\partial f^(0)}{\partial T}\Theta \\ = f^{(0)} - p\frac{\partial f^(0)}{\partial p}\Theta. \end{aligned} \tag{6}$$

Our next task will be collecting terms of similar order in eq. (1).

2.1 Zero-Order Equation

By collecting the zero-order terms in eq. (1) (the terms with no dependence on $\Phi, \Psi,$ or Θ), we get (7):

$$\frac{df}{dt}|_{\text{zero order}} = \frac{\partial f^{(0)}}{\partial t} - Hp\frac{\partial f^{(0)}}{\partial p} = 0. \tag{7}$$

We could assert that $\frac{df}{dt} = 0$ because this is the collision-less equation, and hence $C(x^i, p^i, f) = 0 = \frac{df}{dt}$, but even when collisions are considered, there is no zero-order collision term. We will next rewrite the RHS of eq. (7) using:

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial T}\frac{dT}{dt} = -\frac{dT/dt}{T}p\frac{\partial f^{(0)}}{\partial p}. \tag{8}$$

This gives us that:

$$[-\frac{dT/dt}{T} - \frac{da/dt}{a}]\frac{\partial f^{(0)}}{\partial p} = 0, \tag{9}$$

and thus:

$$\frac{dT}{T} = -\frac{da}{a} \Longleftrightarrow T \propto \frac{1}{a} \tag{10}$$

This is reassuring, as this result coincides with those from other arguments that we have gone over in prior lectures.

2.2 First-Order Equation

By collecting the first-order terms in eq. (1) and substituting in the expansion from eq. (6) for each f, we get an equation for the deviation of the photon temperature from its zero-order value (Θ). This gives us (11):

$$\begin{aligned} \frac{df}{dt}\Big|_{\text{first order}} = -p\frac{\partial}{\partial t}[\frac{\partial f^{(0)}}{\partial p}\Theta] - p\frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i}\frac{\partial f^{(0)}}{\partial p} + Hp\Theta\frac{\partial}{\partial p}[p\frac{\partial f^{(0)}}{\partial p}] \\ -p\frac{\partial f^{(0)}}{\partial p}[\frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}] \end{aligned} \tag{11}$$

We can simplify this equation quite a bit by rewriting the first term on the right hand side as a temperature derivative and making use of the fact that $\frac{\partial f^{(0)}}{\partial T} = -\frac{p}{T}\frac{\partial f^{(0)}}{\partial p}$:

$$\begin{aligned} -p\frac{\partial}{\partial t}[\frac{\partial f^{(0)}}{\partial p}\Theta] &= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial \Theta}{\partial t} - p\Theta\frac{dT}{dt}\frac{\partial^2 f^{(0)}}{\partial T \partial p} \\ &= -p\frac{\partial f^{(0)}}{\partial p}\frac{\partial \Theta}{\partial t} + p\Theta\frac{dT/dt}{T}\frac{\partial}{\partial p}[p\frac{\partial f^{(0)}}{\partial p}] \end{aligned} \tag{12}$$

Substituting eq. (13) back into eq. (11) and combining/cancelling terms gives us the equation that governs our perturbation Θ:

$$\frac{df}{dt}\Big|_{\text{first order}} = -p\frac{\partial f^{(0)}}{\partial p}[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial \Psi}{\partial x^i}] \tag{13}$$

The first two terms account for anisotropies that exist on smaller scales as the universe evolves, and the last two terms account for the effect of gravity.

3 Collision Terms: Compton Scattering

We write the Compton scattering process as:

$$e^-(\vec{q}) + \gamma(\vec{p}) \leftrightarrow e^-(\vec{q}\,') + \gamma(\vec{p}\,') \tag{14}$$

where $\vec{q}, \vec{p}, \vec{q}\,', \vec{p}\,'$ are the corresponding momenta of each particle.

We are interested in the change in the distribution of photons with momentum $\vec{p}$ (magnitude p and direction $\hat{p}$), so we need to sum over all other momenta which affect $f(\vec{p})$, as such, the collision term can be thought of as:

$$C[f(\vec{p})] = \sum_{\vec{q},\vec{q}\,',\vec{p}\,'} |\text{Amplitude}|^2 \{f_e(\vec{q}\,')f(\vec{p}\,') - f_e(\vec{q})f(\vec{p})\} \tag{15}$$

The amplitude is reversible, so it multiplies both the forward and reverse reactions. f_e is the electron distribution function, f is our photon distribution function. The effects of stimulated emission and Pauli blocking are left out, and luckily, to first order, these interactions have no effect.

We will now calculate the actual collision term by integrating over phase space. Note: we will have to multiply by a factor of $\frac{1}{p}$ to correct for the fact that we started with $C(x^i, p^i, f) = \frac{df}{dt}$ and not the derivative with regard to the affine parameter $\frac{df}{d\lambda}$. Keeping these considerations in mind, we can explicitly state the collision term:

$$\begin{aligned} C[f(\vec{p})] = \frac{1}{p} \int \frac{d^3 q}{(2\pi)^3 2E_e(q)} \int \frac{d^3 q'}{(2\pi)^3 2E_e(q')} \int \frac{d^3 p'}{(2\pi)^3 2E(p')} |\mathcal{M}|^2 (2\pi)^4 \\ \times\ \delta^3[\vec{p} + \vec{q} - \vec{p}\,' - \vec{q}\,'] \delta[E(p) + E_e(q) - E(p\,') - E_e(q\,')] \\ \times\ \{f_e(\vec{q}\,') f(\vec{p}\,') - f_e(\vec{q}) f(\vec{p})\} \end{aligned} \tag{16}$$

The delta functions ensure that we only account for collisions that conserve energy and momentum. With the convention $c = 1$, we can write that in the relativistic limit for photons and non-relativistic limit for electrons, we will have:

$$E(p) = p\ ;\ E_e(q) = m_e + \frac{q^2}{(2m_e)} \tag{17}$$

Due to the kinetic energies of electrons needing to be much smaller than their rest energies for the behavior we are interested in to occur, we can replace E_e and $E(p')$ with m_e and p' respectively in the denominators. Making these substitutes and evaluating the $\vec{q}\,'$ integral by using the three-dimensional momentum delta function, we get:

$$\begin{aligned} C[f(\vec{p})] = \frac{\pi}{4m_e^2 p} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 p'}{(2\pi)^3 p'} \delta[p + \frac{q^2}{2m_e} - p' - \frac{(\vec{p} + \vec{q} - \vec{p}\,')^2}{2m_e}] \\ \times\ |\mathcal{M}|^2 \{f_e(\vec{p} + \vec{q} - \vec{p}\,') f(\vec{p}\,') - f_e(\vec{q}) f(\vec{p})\} \end{aligned} \tag{18}$$

We can simplify this further by considering the kinematics of Compton scattering. In the non-relativistic case (the one we are concerned with), very little energy is transferred, and q is much larger than p and p'. Also, the collision is nearly elastic: $p' \simeq p$. Specifically, we can state that:

$$\begin{aligned} E_e(q) - E_e(\vec{q} + \vec{p} - \vec{p}\,') &= \frac{q^2}{2m_e} - \frac{(\vec{q} + \vec{p} - \vec{p}\,')^2}{2m_e}] \\ &\simeq \frac{(\vec{p}\,' - \vec{p}) \cdot \vec{q}}{m_e}. \end{aligned} \tag{19}$$

We will also find that $\vec{p}\,' - \vec{p}$ is of order p, and of order T with regard to ambient temperature. This means that the right hand side of eq. (19) is of order $\frac{Tq}{m_e} \sim T\nu_b$, where ν_b is the baryonic velocity and is very small.

As such, the change in energy due to Compton scattering is of order $T\nu_b$. The typical kinetic energy of the electrons is also of order T, so we can say that the fractional energy change due to a single Compton collision is of order ν_b, and is thus very small. Due to this, we can expand the final electron kinetic energy ($\frac{(\vec{p}+\vec{q}-\vec{p}\,')^2}{2m_e}$) around its zero-order value of $\frac{q^2}{2m_e}$, and can expand the energy delta function as:

$$\delta[p+\frac{q^2}{2m_e}-p'-\frac{(\vec{p}+\vec{q}-\vec{p}\,')^2}{2m_e}] \simeq \delta(p-p')$$
$$+(E_e(q')-E_e(q))\frac{\partial\delta(p+E_e(q)-p'-E_e(q'))}{\partial E_e(q')}|_{E_e(q)=E_e(q')}$$
$$=\delta(p-p')+\frac{(\vec{p}\,'-\vec{p})\cdot\vec{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'}. \tag{20}$$

When we integrate this function over momenta, we will use integration by parts (due to the odd nature of this expression). We will now use eq. (20) along with the fact that $f_e(\vec{q}+\vec{p}-\vec{p}\,') \simeq f_e(\vec{q})$ on eq. (18) to get:

$$C[f(\vec{p})] = \frac{\pi}{4m_e^2 p}\int\frac{d^3q}{(2\pi)^3}f_e(\vec{q})\int\frac{d^3p'}{(2\pi)^3p'}|\mathcal{M}|^2$$
$$\times\{\delta(p-p')+\frac{(\vec{p}\,'-\vec{p})\cdot\vec{q}}{m_e}\frac{\partial\delta(p-p')}{\partial p'}\}\{f(\vec{p}\,')-f(\vec{p})\}. \tag{21}$$

This is the furthest we can go without calculating the amplitude for Compton scattering. This can be computed using Feynman rules. We, however, will proceed by assuming that is is constant and given by:

$$|\mathcal{M}|^2 = 8\pi\sigma_T m_e^2 \tag{22}$$

where $\pi\sigma_T$ is the Thomson cross-section. This is, however, as Dodelson acknowledges, wrong for a few reasons.

Firstly, the amplitude squared will have an angular dependence: $|\mathcal{M}|^2 \propto (1+\cos^2[\hat{p}\cdot\hat{p}'])$. By ignoring this angular dependence, we give up a small amount of accuracy in our final collision term, so for the sake of simplicity, we will ignore it.

The second main issue is due to the fact that the amplitude squared is dependant on polarization: $|\mathcal{M}|^2 \propto |\hat{\epsilon}\cdot\hat{\epsilon}'|^2$ where ϵ and $\hat{\epsilon}'$ are the polarization of the incoming and outgoing photons respectively. This constitutes an issue because we already implicitly summed over these polarizations. Even if we do not explicitly concern ourselves with polarization, the temperature anisotropies are coupled to the polarization field, so we would necessarily have to deal with polarization to talk about temperature anisotropies. All things considered, we will still be ignoring the small effect that this has as well.

Moving forward, while asserting that eq. (22) holds, we will multiply out the terms in the brackets in eq. (21) while keeping only terms that are first order with regard to energy transfer. We will also make use of the fact that the integral over $\vec{q}$ will give a factor of n_e and $n_e\vec{\nu_b}$ for $\frac{\vec{q}}{m_e}$. Taking this into account and breaking up $f(\vec{p}\,')-f(\vec{p})$ into a zero-order piece (which will not contribute when multiplying by $\delta(p-p')$ and a first-order term which can be ignored when multiplying by the velocity term:

$$
\begin{aligned}
C[f(\vec{p})] = {} & \frac{2\pi^2 n_e \sigma_T}{p} \int \frac{d^3p'}{(2\pi)^3 p'} |\{\delta(p-p') + (\vec{p}-\vec{p}\,') \cdot \vec{\nu}_b \frac{\partial \delta(p-p')}{\partial p'}\} \\
& \times \{f(\vec{p}\,') - f(\vec{p}) - p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}') + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p})\} \\
= {} & \frac{n_e \sigma_T}{4\pi p} \int_0^\infty dp' p' \int d\Omega' [\delta(\vec{p}-\vec{p}\,')(-p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}') + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p})) \\
& + (\vec{p}-\vec{p}\,') \cdot \vec{\nu}_b \frac{\partial \delta(p-p')}{\partial p'} (f^{(0)}(p') - f^{(0)}(p))]
\end{aligned}
\tag{23}
$$

where Ω' is the solid angle spanned by the unit vector $\hat{p}'$.

There are only two terms that depend on $\hat{p}'$ and thus affect the integral over Ω': the $\hat{p}' \cdot \vec{\nu}_b$ term and the distribution function $\Theta(\hat{p}')$. Due to $\vec{\nu}_b$ being a constant vector, the first of these terms integrates to zero. To deal with the second term, we will introduce the following substitution:

$$
\Theta_0(\vec{x}, t) \equiv \frac{1}{4\pi} \int d\Omega' \Theta(\hat{p}', \vec{x}, t). \tag{24}
$$

Due to us integrating over all directions, Θ_0 has no dependence on $\hat{p}'$. Thus, it is the monopole part of the perturbation. We cannot absorb this into the zero-order temperature because it will not be constant over all space. Thus, this perturbation represents the deviation of the monopole at a given point in space from its average over all space. We can now integrate over the solid angle to get:

$$
\begin{aligned}
C[f(\vec{p})] = {} & \frac{n_e \sigma_T}{p} \int_0^\infty dp' p' [\delta(\vec{p}-\vec{p}\,')(-p' \frac{\partial f^{(0)}}{\partial p'} \Theta(\hat{p}') + p \frac{\partial f^{(0)}}{\partial p} \Theta(\hat{p})) \\
& + \vec{p} \cdot \vec{\nu}_b \frac{\partial \delta(p-p')}{\partial p'} (f^{(0)}(p') - f^{(0)}(p))]
\end{aligned}
\tag{25}
$$

We finish by computing the p' integral by using the delta function and integration by parts to get:

$$
C[f(\vec{p})] = -p \frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \vec{\nu}_b] \tag{26}
$$

From (finally) knowing this collision term, we can begin to predict the effect of Compton scattering on the photon distribution.

In the case of no bulk velocity ($\nu_b = 0$), the collision terms drive Θ to Θ_0. This means that all higher order moments will disappear. Heuristically, strong scattering will mean that photons are only moving freely over small paths. This means that any photons arriving at a given point will have recently been scattered off of another, very close electron. These electrons, due to their close proximity, will have temperatures that are very similar. This will mean that, from a point of observation, photons will have the same temperature regardless of the direction they came from. This means that temperature will be uniform in the monopole distribution.

If, however, the electrons do carry a bulk velocity, the dipole moment will not vanish. This dipole will depend on the amplitude and direction of the electron velocity. Even in this case, though, only the dipole and monopole will remain. That is to say, photons will behave like a fluid, and in the case of strong scattering, photons and electrons will behave as a single fluid.

4 The Boltzmann Equation for Photons

After much work, we can finally write the Boltzmann equation for photons. We do this by equating eqs. (13) & (26) to get:

$$\frac{\partial\Theta}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{\nu}_b] \tag{27}$$

From here, we are able to work produce a linear equation for the perturbation of the photon distribution.

We begin to do this by reintroducing the conformal time, η, as the time variable. In terms of η, and representing derivatives with regard to it by dots on our functions, the Boltzmann equation becomes:

$$\dot{\Theta} + \frac{\hat{p}^i}{a}\frac{\partial\Theta}{\partial x^i} + \dot{\Phi} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T[\Theta_0 - \Theta(\hat{p}) + \hat{p}\cdot\vec{\nu}_b]. \tag{28}$$

This is a partial differential equation that couples our perturbation, Θ to $\Phi, \Psi,$ and $\vec{\nu}_b$. We will proceed by making use of Fourier transforms. Our convention will be:

$$\Theta(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}}\tilde{\Theta}(\vec{k}) \tag{29}$$

Next, we will define the cosine of the angle between the wave-vector $\vec{k}$ and the photon's direction of travel $\hat{p}$ to be $/mu$ as such:

$$\mu \equiv \frac{\vec{k}\cdot\hat{p}}{k}. \tag{30}$$

We will typically assume that the velocity points in the same direction as $\vec{k}$, so normally, we will have $\tilde{\vec{\nu}}_b\cdot\hat{p} = \tilde{\nu}_b\mu$. Next, we will define the optical depth τ:

$$\tau(\eta) \equiv \int_\eta^{\eta_0} d\eta' n_e\sigma_T a. \tag{31}$$

Notice that the limits of integration lead to:

$$\dot{\tau} \equiv \frac{d\tau}{d\eta} = -n_e\sigma_T a. \tag{32}$$

Finally, we can write:

$$\dot{\tilde{\Theta}} + ik\mu\tilde{\Theta} + \dot{\tilde{\Phi}} + ik\mu\tilde{\Psi} = -\dot{\tau}[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{\nu}_b], \tag{33}$$

which represents an uncoupled system of ordinary differential equations. This concludes our discussion of the Boltzmann equation for photons.

References

[1] Dodelson, S. (2003). *Modern cosmology (pp. 84-113).* San Diego, Calif.: Academic Press.

[2] Ellis, G., Matravers, D., & Treciokas, R. (1983).*Anisotropic solutions of the Einstein-Boltzmann equations: I. General formalism.* Annals Of Physics, 150(2), 455-486.