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QUANTUM FIELD THEORY

Jean Weill

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Professor S.G. Rajeev
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1 Introduction

We can imagine that this complicated array of moving things which constitutes the world is something like a great chess game being played by the gods, and we are observers of the game. We do not know what the rules of the game are; all we are allowed to do is to watch the playing.

Richard Feynman (1918-1988)

The objective of this course was to introduce ourselves to the gigantic and fascinating modern domain that is quantum field theory. Quantum field theory allows one to work with relativistic quantum system and predicts groundbreaking events that have been proved to be right over and over again for the past 50 years.

In the first section we will learn how to quantized the electromagnetic field through a mix of classical arguments, Fourier series, and discrete quantum analogies. In the second section we will generalize this approach to a continuous system by taking the fields at each point in space as the dynamical variables and quantize them directly.

2 Photons and the Electromagnetic Field

2.1 The Electromagnetic Field in the Absence of Charges

Consider Maxwell's equations in the absence of charges, i.e., where the charge density $\rho = 0$ and the current density $\mathbf{j} = \mathbf{0}$. I will let c=1 for the entire paper (but not $\hbar$, I like $\hbar$).

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned} \tag{1}$$

The divergence of the magnetic field is 0, so $\mathbf{B}$ can be written as the curl of a vector field $\mathbf{A}$ since $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ for any arbitrary field $\mathbf{A}$. This vector field is of course not unique since the divergence of the gradient of an arbitrary scalar field λ is 0. Thus, we can make the following transformation

$$\mathbf{A}' = \mathbf{A} - \nabla\lambda \tag{2}$$

You have to keep in mind that $\mathbf{A}$ is just a mathematical construct, it doesn't have any physical meaning. We do not observe it, what we do observe are the electric and magnetic field. So, for example, if you find some causality issues with $\mathbf{A}$, do not worry, they will vanish when you take the curl of it. Plugging in for $\mathbf{B}$ in term of $\mathbf{A}$ in the last Maxwell's equation, we get

$$\nabla \times \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{A}}{\partial t}\right) \tag{3}$$

Therefore, since the curl of the gradient of an arbitrary field V is 0, we get a new expression for $\mathbf{E}$.

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{4}$$

Looking at how $\mathbf{E}$ transform when we change $\mathbf{A}$ according to (2), we see that in order for $\mathbf{E}$ to remain invariant under such a transformation, we need

$$V' = V + \frac{\partial \lambda}{\partial t} \tag{5}$$

The transformations (2) and (5) are called the gauge transformations, and leave the electric and magnetic field invariant. It is crucial for a theory formulated in terms of potentials to be gauge-invariant since what we actually observe should not be alter by such mathematical constructs. Also, the fact that a transformation leaves a field invariant should ring a bell. Noether's theorem states that some quantity must therefore be conserved.

Now, let's focus on the first two Maxwell's equations. Using what we just derived, we get

$$\partial_\mu \partial^\mu \mathbf{A} + \nabla \left(\frac{\partial V}{\partial t} + \nabla \cdot \mathbf{A} \right) = 0 \tag{6}$$

where

$$\partial_\mu \partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{7}$$

Remember that both the charge density and the current density are 0, and using the first of Maxwell's equations, we get that $\nabla^2 V = 0$. Since we want V to vanish at infinity, the only solution to this equation is $V = 0$, therefore, we have that $\nabla V = 0$. Finally, choosing the Coulomb gauge where $\nabla \cdot \mathbf{A} = 0$, equation (6) becomes

$$\partial_\mu \partial^\mu \mathbf{A} = 0 \tag{8}$$

which is exactly the wave equation! The most general solution is

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \tag{9}$$

where $\mathbf{k}$ is the wave number and ω is the angular frequency. Since $\nabla \cdot \mathbf{A} = 0$, we have that $\mathbf{k} \cdot \mathbf{A} = 0$, i.e., $\mathbf{A}$ is perpendicular to the direction of propagation $\mathbf{k}$ of the wave. This type of wave is called a transverse wave. The solution to (8) are the transverse electromagnetic waves in free space, often called the radiation field. In order to express $\mathbf{A}$ in term of an infinite but countable number of degrees of freedom we have to make some approximation so that Fourier series tricks analogous to the basic vibrating string can be applied. We first need an enclosure with some periodic boundary conditions. Let's take a large cube of side L and require that $\mathbf{A}(0, y, z, t) = \mathbf{A}(L, y, z, t)$, $\mathbf{A}(x, 0, z, t) = \mathbf{A}(x, L, z, t)$, $\mathbf{A}(x, y, 0, t) = \mathbf{A}(x, y, L, t)$. We can express $\mathbf{A}$ as a Fourier series.

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\mathbf{k}} \sum_{r} \left(\frac{\hbar}{2V\omega_{\mathbf{k}}} \right)^{\frac{1}{2}} \boldsymbol{\varepsilon}_r(\mathbf{k}) \left(a_r(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{x}} + a_r^*(\mathbf{k}, t) e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \tag{10}$$

where $\mathbf{k} = \frac{2\pi}{L}(n_1, n_2, n_3)$, with $n_1, n_2, n_3 = 0, 1, -1, ...$; $\omega_{\mathbf{k}} = \left(\mathbf{k}^2\right)^{\frac{1}{2}}$; $\boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \boldsymbol{\varepsilon}_s(\mathbf{k}) = \delta_{rs}$, and $\boldsymbol{\varepsilon}_r(\mathbf{k}) \cdot \mathbf{k} = 0$, with $r, s = 1, 2$ being the polarization states for each $\mathbf{k}$. We can now solve for the energy of the field in terms of the amplitudes since we know how to express $\mathbf{E}$ and $\mathbf{B}$ in term of $\mathbf{A}$, we get

$$\begin{aligned} H_{rad} &= \frac{1}{2} \int \left(\mathbf{E}^2 + \mathbf{B}^2 \right) d^3\mathbf{x} \\ &= \sum_{\mathbf{k}} \sum_{r} \hbar \omega_{\mathbf{k}} a_r^*(\mathbf{k}) a_r(\mathbf{k}) \end{aligned} \tag{11}$$

This equation should look a bit familiar. What if we alter it "slightly"?

$$H_{rad} = \sum_{\mathbf{k}} \left(\hbar \omega_{\mathbf{k}} \sum_{r} \left(a_r^*(\mathbf{k}) a_r(\mathbf{k}) + \frac{1}{2} \right) \right) \tag{12}$$

What is inside the first parenthesis is exactly what the quantum oscillator's Hamiltonian looks like, one for each mode of the radiation, with number operator $N_r(\mathbf{k}) = a_r^\dagger(\mathbf{k})a_r(\mathbf{k})$. This pushes us to copy the harmonic oscillator formalism, which allows us to quantize the field. We get the following commutation relations

$$\begin{aligned} \left[a_r(\mathbf{k}), a_s^\dagger(\mathbf{k}')\right] &= \delta_{rs}\delta_{\mathbf{k}\mathbf{k}'} \\ \left[a_r(\mathbf{k}), a_s(\mathbf{k}')\right] &= \left[a_r^\dagger(\mathbf{k}), a_s^\dagger(\mathbf{k}')\right] = 0 \end{aligned} \tag{13}$$

When we changed the Hamiltonian to make it resemble the harmonic oscillator one, we are technically adding an infinite constant. However, this has no physical significance since we can eliminate it by shifting the zero of the energy scale to coincide with the vacuum state $|0\rangle$. So we can actually get rid of the constant all together in the expression for the Hamiltonian itself.

We can slip up equation (10) in two parts: one containing only a operators, and the other only $a^\dagger$ operators. Thinking back to the basic quantum mechanical meaning of these operators, we can think of the a operators as the absorption operators, and the $a^\dagger$ as the creation operators. Summarizing this, we get

$$\mathbf{A}(\mathbf{x}, t) = \mathbf{A}^+(\mathbf{x}, t) + \mathbf{A}^-(\mathbf{x}, t) \tag{14}$$

where

$$\mathbf{A}^+(\mathbf{x}, t) = \sum_{\mathbf{k}}\sum_{r}\left(\frac{\hbar}{2V\omega_{\mathbf{k}}}\right)^{\frac{1}{2}} \boldsymbol{\varepsilon}_r(\mathbf{k})a_r(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} \tag{15}$$

and

$$\mathbf{A}^-(\mathbf{x}, t) = \sum_{\mathbf{k}}\sum_{r}\left(\frac{\hbar}{2V\omega_{\mathbf{k}}}\right)^{\frac{1}{2}} \boldsymbol{\varepsilon}_r(\mathbf{k})a_r^\dagger(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} \tag{16}$$

3 Lagrangian Field Theory

3.1 Classical Lagrangian Field Theory

We now consider the four-dimensional space-time continuum with action integral

$$S(\Omega) = \int_\Omega \mathcal{L}(\phi_r, \phi_{r,\alpha})d^4x \tag{17}$$

where Ω is an arbitrary region of the four-dimensional space-time continuum, $\mathcal{L}$ is the Lagrangian density, ϕ_r is one of the several fields the system requires, and $\phi_{r,\alpha} = \partial_\alpha\phi_r$. We want to recover the Euler-Lagrange equations for this higher dimensional case. We can try to apply the same type of variational approach. We want to apply a small perturbation to our field, one that will leave the action unchanged. We also want the "end points" fixed, which in this case is the surface $\Gamma(\Omega)$ which bounds Ω. In a formal mathematical way, this is equivalent to

$$\phi_r(x) \to \phi_r(x) + \delta\phi_r(x)$$

$$\delta\phi_r(x) = 0 \text{ on } \Gamma(\Omega)$$

Now, we want to exploit the $\delta S(\Omega) = 0$ constraint. We can do this by first focusing on $\delta\mathcal{L}$ since $\delta S(\Omega) = \int_\Omega \delta\mathcal{L}d^4x$.

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\delta\phi_{r,\alpha} \\
&= \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\partial_\alpha\left(\delta\phi_r\right) \\
&= \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \left[\partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\delta\phi_r\right) - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\right)\delta\phi_r\right] \\
&= \left[\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\right)\right]\delta\phi_r + \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\delta\phi_r\right)
\end{aligned} \tag{18}
$$

Using Gauss's divergence theorem and the fact that $\delta\phi_r = 0$ on Γ, we can see that when we integrate $\delta\mathcal{L}$ the second term vanishes. Therefore, we are left with

$$
\delta S(\Omega) = \int_\Omega \delta\phi_r \left[\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\right)\right] d^4x \tag{19}
$$

Since $\delta S(\Omega) = 0$ for an arbitrary regions Ω and arbitrary variations $\delta\phi_r$, we get the following four-dimensional space-time continuum Euler-Lagrange equations

$$
\frac{\partial\mathcal{L}}{\partial\phi_r} - \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\right) = 0; \alpha = 0,1,2,3; r = 1,...,N \tag{20}
$$

In our approach to derive the equation of motions for the fields, we considered the fields at each point in space as dynamical variables. Now, we want to quantize them. The idea is to generalizes the classical mechanics of a system of particles, and its quantization, to a continuous system, i.e., to fields. This is not simple since in this case our system as a continuously (uncountable) infinite number of degrees of freedom, corresponding to the values of the fields ϕ_r, considered as functions of time, at each point of the three dimensional spatial space. In order to get the continuous case, we will have to make some approximations to end up with a system with a countable number of degrees of freedoms. Then, we will be able to apply the quantization techniques from the well known discrete case. Finally, the continuous case will be found as a limit of the discrete case. The details of these calculations are quite straight forward, the only subtle element is the first part, i.e., how to get a countable number of degrees of freedom. To do this, we have to decompose the three dimensional spatial space into small cells of equal volume $\delta\mathbf{x}_i$. Since these volumes are infinitesimal, we can approximate the values of the fields within each one of these cells by each cell center's value. We therefore end up with a discrete (infinite but countable) set of generalized coordinates. The discrete cases should be familiar, thus I only want to describe how the common discrete equations for each one of these generalized coordinates transform as we take the limit $\delta\mathbf{x}_i \to 0$.

$$
L(t) = \sum_i \delta x_i \mathcal{L}_i\left(\phi_r(i,t), \partial_0\phi_r(i,t), \phi_r(i',t)\right) \to \int \mathcal{L}(\phi_r, \phi_{r,\alpha}) d^3\mathbf{x} \tag{21}
$$

The discrete momenta conjugate to $\phi_r(i,t)$ is defined in the usual way

$$
p_r(i,t) = \frac{\partial L}{\partial\left(\partial_0\phi_r(i,t)\right)} = \frac{\partial L}{\partial\dot{\phi}_r(i,t)} = \pi_r(i,t)\delta\mathbf{x}_i \tag{22}
$$

$$
H = \sum_i \delta x_i \left(\pi_r(i,t)\partial_0\phi_r(i,t) - \mathcal{L}_i\right) \to \int \left(\pi_r(x)\partial_0\phi_r(x) - \mathcal{L}\right) d^3\mathbf{x} = \int \mathcal{H} d^3\mathbf{x} \tag{23}
$$

where $\mathcal{H}$ is called the Hamiltonian density, and the

$$
\pi_r(x) = \frac{\partial\mathcal{L}}{\partial\left(\partial_0\phi_r\right)} = \frac{\partial\mathcal{L}}{\partial\dot{\phi}_r} \tag{24}
$$

are the fields conjugate to $\phi_r(x)$. The two integrals in equation (21) and (22) are over all space, at time t. With our Lagrangian density which does not depend explicitly on time, the Hamiltonian H is of course constant in time.

3.2 Quantized Lagrangian Field Theory

In order to quantize the field, we want to adopt a similar approach as the one used in the previous section to go from the known discrete case to the continuous one. The discrete canonical relations are the well known

$$[\phi_r(j,t), p_s(j',t)] = i\hbar\delta_{rs}\delta_{jj'} \tag{25}$$

$$[\phi_r(j,t), \phi_s(j',t)] = [p_r(j,t), p_s(j',t)] = 0 \tag{26}$$

The first Kronecker delta in equation (25) is the same as the one we expect from quantum mechanics. The second one gives the logical constraint that they have to be in the same cell.

Using equation (22), we can express these relations in term of the fields conjugate to $\phi_r(j,t)$, giving

$$[\phi_r(j,t), \pi_s(j',t)] = i\hbar\frac{\delta_{rs}\delta_{jj'}}{\delta\mathbf{x}_j} \tag{27}$$

$$[\phi_r(j,t), \phi_s(j',t)] = [\pi_r(j,t), \pi_s(j',t)] = 0 \tag{28}$$

Letting $\delta\mathbf{x}_j \to 0$, we get the continuous canonical relations, i.e., the commutation relations for the fields:

$$[\phi_r(\mathbf{x},t), \pi_s(\mathbf{x'},t)] = i\hbar\delta_{rs}\delta(\mathbf{x}-\mathbf{x'}) \tag{29}$$

$$[\phi_r(\mathbf{x},t), \phi_s(\mathbf{x'},t)] = [\pi_r(\mathbf{x},t), \pi_s(\mathbf{x'},t)] = 0 \tag{30}$$

where $\mathbf{x}$ and $\mathbf{x'}$ are from the j^{th} and j'^{th} cell, respectively. Note that the canonical commutation relations involve the fields at the same time; they are equal-time commutation relations.

3.3 Symmetries and Conservation Laws

This entire section revolves around one of the most powerful idea in physics: for any symmetry there is a conserved quantity. This fact allows one to simplify seemingly difficult equations, and find hidden beauty in nature as we will very soon see. We now have the tool to derive the theorem behind this relation: Noether's theorem. First we have to understand what a symmetry is. The mathematician and theoretical physicist Weyl defined symmetry in a very interesting way, he said: "a thing is symmetrical if there is something you can do to it so that after doing it, it looks the same as it did before". So this is exactly what we are going to do to our Lagrangian. We will apply the transformation

$$\phi_r(x) \to \phi_r(x) + \delta\phi_r(x) \tag{31}$$

and this what happens if we require the Lagrangian to stay invariant under such a transformation, i.e, $\mathcal{L} \to \mathcal{L} + \delta\mathcal{L} = \mathcal{L}$. Which is equivalent to saying

$$\delta\mathcal{L} = 0 \tag{32}$$

but, under the above transformation, we have

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_r}\delta\phi_r + \frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\delta\phi_{r,\alpha} \tag{33}$$

which becomes, using (20),

$$\delta\mathcal{L} = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial\mathcal{L}}{\partial\phi_{r,\alpha}}\delta\phi_r\right) \tag{34}$$

which, using (32), give us the conserved quantity so desired

$$f^\alpha = \frac{\partial \mathcal{L}}{\partial \phi_{r,\alpha}} \delta\phi_r \tag{35}$$

We can get another very useful conserved quantity from this principle using some smart integration tricks. Let

$$F^\alpha = \int_\Omega d^3\mathbf{x} f^\alpha(x) \tag{36}$$

where Ω is all space. Using the what we just derived, i.e., $\partial_\alpha f^\alpha = 0$, we get

$$\begin{aligned} \int_\Omega d^3\mathbf{x}\partial_\alpha f^\alpha &= 0 \\ \int_\Omega d^3\mathbf{x}\partial_0 f^0 &= -\int_\Omega d^3\mathbf{x}\partial_j f^j \\ \frac{dF^0}{dt} &= -\int_\Omega d^3\mathbf{x}\partial_j f^j \end{aligned} \tag{37}$$

The last integral on the right is nothing less than the volume integral of the divergence of $\mathbf{f}$ over all space, thus, using Gauss's divergence theorem, we get.

$$\int_\Omega d^3\mathbf{x}\partial_j f^j = \int_\Omega d^3\mathbf{x}\nabla\cdot\mathbf{f} = \int_S d^2\mathbf{x}\cdot\mathbf{f} = 0 \tag{38}$$

So we get the other very useful conserved quantity

$$F^0 = \int_\Omega d^3\mathbf{x} f^0(x) \tag{39}$$

Using (24), we can rewrite this equation in a slightly more enlightening way, since

$$f^0 = \frac{\partial \mathcal{L}}{\partial \phi_{r,0}} \delta\phi_r = c\pi_r\delta\phi_r \tag{40}$$

where the π_r are the fields conjugate to ϕ_r. Then, (39) becomes

$$F^0 = \int_\Omega d^3\mathbf{x}\pi_r(x)\delta\phi_r(x) \tag{41}$$

Now that we have all the theory dawn concerning this symmetry implying conserved quantity business, let's work out an example that is of prime interest in electrodynamics. Suppose you are given a Lagrangian that is dependent on the independent fields ϕ_r and ϕ_r^*, where the $*$ means complex conjugate. Furthermore, suppose that your Lagrangian is invariant under the following infinitesimal rotation

$$\begin{aligned} \phi_r &\to e^{i\epsilon}\phi_r = \phi_r + \delta\phi_r \\ \phi_r^* &\to e^{-i\epsilon}\phi_r = \phi_r^* + \delta\phi_r^* \end{aligned} \tag{42}$$

Then, using the Taylor series approximation for e and ignoring term of $\mathcal{O}(\epsilon^2)$, we get

$$\begin{aligned} (1+i\epsilon)\phi_r &= \phi_r + \delta\phi_r \\ (1-i\epsilon)\phi_r^* &= \phi_r^* + \delta\phi_r^* \end{aligned} \tag{43}$$

and so,

$$\begin{aligned} i\epsilon\phi_r &= \delta\phi_r \\ -i\epsilon\phi_r^* &= \delta\phi_r^* \end{aligned} \tag{44}$$

Plugging this result into (40), we get the conserved quantity

$$f^0 = \pi_r i\epsilon\phi_r - \pi_r^* i\epsilon\phi_r^* = j \tag{45}$$

Those familiar with electrodynamics might recognize this quantity to be nothing less than the conserved current from Maxwell's equation when the E&M Lagrangian is used. The other conserved quantity is

$$F^0 = i\epsilon \int d^3\mathbf{x}\,(\pi_r(x)\delta\phi_r(x) - \pi_r^*(x)\phi_r^*(x)) \tag{46}$$

Since F^0 is a conserved quantity, i.e., equals to a constant; we can multiply it by another constant and still have a conserved quantity: Q

$$Q = -\frac{iq}{\hbar} \int d^3\mathbf{x}\,(\pi_r(x)\delta\phi_r(x) - \pi_r^*(x)\phi_r^*(x)) \tag{47}$$

where q is a constant that, as you might have guessed considering the parallel with E&M discussed above, will turn out to be directly linked to the electric charge of the particles represented by the fields. Q can therefore be interpreted as the charge operator. In fact, ϕ_r and ϕ_r^* are linear in creation and absorption operators, with ϕ_r absorbing particles of charge $+q$ or creating particles of charge $-q$, while ϕ_r^* absorbs particles of charge $-q$ or creates particles of charge q.

4 Conclusion

Mathematics is a game where mathematicians invent the rules. Physics is a game where the rules are given to us by nature. What is interesting is that the rules of nature appear to be in the same mathematical rules as the mathematicians have concocted.

P.A.M Dirac (1902-1984)

The above sections are the foundations of quantum field theory. They allow one to have the necessary tools to quantize a general classical field and therefore understand how bosons behave when thought of as excitations of their corresponding bosonic fields.

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