Lowest-Order $e^+e^- \to l^+l^-$ Processes in Quantum Electrodynamics

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1 Introduction

In this short paper, we will demonstrate some of the simplest calculations in quantum electrodynamics (QED), leading to the lowest-order experimental predictions for cross-sections.

In particular, we will first define what (differential) cross-sections are. Then, we will discuss spin/polarization sums and averages, which are important for experimental predictions. This arises because our experimental set-ups are agnostic to the spin/polarization states of the external particles. Afterwards, we will calculate in detail the cross-sections for one of the simplest and the most common processes in QED: $e^+e^- \to l^+l^-$. We treat the case where $l = e$ separately, since that requires much more calculations.

2 QED Cross-Section, Spin, and Polarization

Let us label the initial and final states of a quantum system as $|i\rangle$ and $|f\rangle$ respectively. Then, the *transition probability* from $|i\rangle$ to $|f\rangle$ is given by

$$\mathcal{P}(i \to f) = |S_{fi}|^2 , \tag{1}$$

where the S-matrix element S_{fi} is defined as

$$S_{fi} \equiv \langle f| S |i\rangle \tag{2}$$

and signifies the *transition amplitude* from a state $|i\rangle$ to a state $|f\rangle$ within a time duration T and a volume V, not explicitly included in eq. 1.

More specifically, consider a process where we begin with an initial state i with two particles whose 4-momenta are $p_i = (E_i, \boldsymbol{p_i})$, $i = 1, 2$, and end up with a final state f with N particles whose 4-momenta are $p'_f = \left(E'_f, \boldsymbol{p'_f}\right)$, $f = 1, 2, ..., N$. In this paper, we will only discuss QED processes, and hence the particles involved are leptons and photons only.

Then, if we specify the time T and the spatial volume V under consideration, the S-matrix element is given by

$$\begin{aligned} S_{fi,TV} = \delta_{fi} + \delta_{TV} &\left(\sum_f p'_f - \sum_i p_i\right) \\ &\times \prod_i \left(\frac{1}{2VE_i}\right)^{1/2} \prod_f \left(\frac{1}{2VE'_f}\right)^{1/2} \prod_l (2m_l)^{1/2} \mathcal{M} , \end{aligned}$$

where the index l runs over all external leptons in the process and δ_{TV} is defined as

$$\delta_{TV}\left(\sum_f p'_f - \sum_i p_i\right) \equiv \int_{-T/2}^{T/2} \mathrm{d}t \int_V \mathrm{d}^3\boldsymbol{x} \, \exp\left[ix^\mu \left(\sum_f p'_f - \sum_i p_i\right)_\mu\right] .$$

$\mathcal{M}$ is called the *invariant amplitude* because this quantity is Lorentz-invariant (or the Feynman amplitude); it is determined by the relevant Feynman diagrams for specific processes under consideration.

Now, suppose T and V are very large. Then,

$$\delta_{TV}\left(\sum_f p'_f - \sum_i p_i\right) \simeq (2\pi)^4 \, \delta^{(4)}\left(\sum_f p'_f - \sum_i p_i\right)$$

due to Fourier transform, and

$$\left[\delta_{TV}\left(\sum_f p'_f - \sum_i p_i\right)\right]^2 = TV \, (2\pi)^4 \, \delta^{(4)}\left(\sum_f p'_f - \sum_i p_i\right) .$$

Then, the transition probability per unit time is

$$
\begin{aligned}
w &= \frac{\left|S_{fi,TV}\right|^2}{T} \\
&= V\,(2\pi)^4\,\delta^{(4)}\left(\sum_f p'_f - \sum_i p_i\right) \\
&\qquad\qquad \times \left(\prod_i \frac{1}{2VE_i}\right)\left(\prod_f \frac{1}{2VE'_f}\right)\left(\prod_l 2m_l\right)\left|\mathcal{M}\right|^2 .
\end{aligned}
\tag{3}
$$

This result only holds for one exact final state f. Practically, we are interested in the transition rate to a set of final states within some momenta range $\left(\boldsymbol{p'_f}, \boldsymbol{p'_f} + \mathrm{d}^3\boldsymbol{p'_f}\right)$. This gives eq. 3 an additional factor of $\prod_f \frac{V\mathrm{d}^3\boldsymbol{p'_f}}{(2\pi)^3}$. Furthermore, it is useful to normalize the transition rate to one scattering/colliding center (recall that we have only 2 particles initially) in the volume and unit incident flux; this requires an additional factor of V/v_{rel}, where v_{rel} is the relative velocity of the two initial particles. Then, we obtain a quantity called the *differential cross-section* which equals

$$
\begin{aligned}
\mathrm{d}\sigma &= w\frac{V}{v_{\mathrm{rel}}}\prod_f \frac{V\mathrm{d}^3\boldsymbol{p'_f}}{(2\pi)^3} \\
&= (2\pi)^4\,\delta^{(4)}\left(\sum_f p'_f - \sum_i p_i\right)\frac{1}{4E_1E_2v_{\mathrm{rel}}}\left(\prod_l 2m_l\right)\left(\prod_f \frac{\mathrm{d}^3\boldsymbol{p'_f}}{(2\pi)^3\,2E'_f}\right)\left|\mathcal{M}\right|^2 .
\end{aligned}
\tag{4}
$$

Note that the two factors of V from the initial-state product are canceled with one factor in w and another in the normalization factor V/v_{rel}.

Assuming that the two initial particles are moving co-linearly, one particularly useful frame of reference is the *center-of-momentum frame* (CoM)—it is also often called the center-of-mass frame, which is a misnomer. In this frame, $\boldsymbol{p_1} = -\boldsymbol{p_2}$, so the relative velocity of the two initial particles is

$$
v_{\mathrm{rel}} = \frac{|\boldsymbol{p_1}|}{E_1} + \frac{|\boldsymbol{p_2}|}{E_2} = \frac{E_1 + E_2}{E_1E_2}|\boldsymbol{p_1}| .
\tag{5}
$$

Now, we consider the specific case in which there are only two particles in the final state as well. Then, eq. 4 becomes

$$
\mathrm{d}\sigma = f(p'_1, p'_2)\,\delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)\,\mathrm{d}^3\boldsymbol{p'_1}\,\mathrm{d}^3\boldsymbol{p'_2} ,
\tag{6}
$$

where

$$
f(p'_1, p'_2) \equiv \frac{\left|\mathcal{M}\right|^2}{64\pi^2 v_{\mathrm{rel}}E_1E_2E'_1E'_2}\prod_l (2m_l) .
\tag{7}
$$

However, due to energy- and momentum-conservations, the final state 4-momenta are not truly independent. For instance, in the CoM frame, $\boldsymbol{p_1} = -\boldsymbol{p_2}$, and $\boldsymbol{p'_1} = -\boldsymbol{p'_2}$.

To write the cross-section in more useful independent variables, we first integrate eq. 6 over $\boldsymbol{p'_2}$ and obtain

$$\mathrm{d}\sigma = f(p'_1, p'_2)\, \delta(E'_1 + E'_2 - E_1 - E_2) \left|\boldsymbol{p'_1}\right|^2 \mathrm{d}\left|\boldsymbol{p'_1}\right| \mathrm{d}\boldsymbol{\Omega'_1} \,,$$

where we now have the momentum-conservation condition $\boldsymbol{p'_2} = \boldsymbol{p_1} + \boldsymbol{p_2} - \boldsymbol{p'_1}$. Integrating over $\left|\boldsymbol{p'_1}\right|$, we obtain

$$\mathrm{d}\sigma = f(p'_1, p'_2) \left|\boldsymbol{p'_1}\right|^2 \mathrm{d}\boldsymbol{\Omega'_1} \left[\frac{\partial\left(E'_1 + E'_2 - E_1 - E_2\right)}{\partial\left|\boldsymbol{p'_1}\right|}\right]^{-1} , \tag{8}$$

where we have the momentum- and energy-conservation condition $p'_2 = p_1 + p_2 - p'_1$.

As mentioned, in the CoM frame, the final-state momenta are not independent: $\boldsymbol{p'_1} = -\boldsymbol{p'_2}$. Hence,

$$\begin{aligned}
\frac{\partial\left(E'_1 + E'_2 - E_1 - E_2\right)}{\partial\left|\boldsymbol{p'_1}\right|} &= \frac{\partial\left(E'_1 + E'_2\right)}{\partial\left|\boldsymbol{p'_1}\right|} \\
&= \frac{\left|\boldsymbol{p'_1}\right|}{E'_1} + \frac{\left|\boldsymbol{p'_1}\right|}{E'_2} \\
&= \frac{E_1 + E_2}{E'_1 E'_2}\left|\boldsymbol{p'_1}\right| \,,
\end{aligned} \tag{9}$$

where we have used the CoM condition, $\left|\boldsymbol{p'_1}\right| = \left|\boldsymbol{p'_2}\right|$, and the energy-conservation condition, $E'_1 + E'_2 = E_1 + E_2$.

Substituting eqs. 5, 7, and 9 into eq. 8, we obtain the CoM differential cross-section

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\boldsymbol{\Omega'_1}}\right)_{\mathrm{CoM}} = \frac{1}{64\pi^2\left(E_1 + E_2\right)^2} \frac{\left|\boldsymbol{p'_1}\right|}{\left|\boldsymbol{p_1}\right|} \left(\prod_l 2m_l\right) \left|\mathcal{M}\right|^2 . \tag{10}$$

However, even this result is too 'specific' for empirical purposes. That is, the initial and the final states of the particles are still specified completely. In particular, the spin and the polarization states of the relevant leptons and photons must be specified; these are included within the invariant amplitude $\mathcal{M}$. However, in most (if not all) practical experimental settings, the incoming particles are typically not polarized, and the polarizations of the outgoing particles are not detected either—hereon, the term polarization will be used loosely to refer the spins of leptons as well as the photon polarizations.

Hence, in order to obtain practical cross-sections for experimental predictions, we need to *average* over the different polarizations of the incoming particles, to which we are agnostic. Similarly, we need to sum up the different outgoing particles' polarization states; since processes with different final polarization states are disjoint, the probabilities can be simply added.

For instance, consider a QED process that has one initial-state lepton with 4-momentum p & spin r and one final-state lepton with 4-momentum p' & spin s. Then, its invariant amplitude is of the form

$$\mathcal{M}_{rs} = \bar{u}_s(p')\Gamma u_r(p)\,, \tag{11}$$

where u and $\bar{u}$ are the positive-energy Dirac spinors and Γ is a 4-by-4 matrix made out of γ-matrices. The specific form of Γ depends on the specific QED process under consideration.

As mentioned, eq. 11 is too specific and limited to fixed spin states r, s. If we want to obtain the experimentally useful *unpolarized* cross-section, we need to average over r and sum over s. Hence, we get a cross-section proportional to the quantity

$$X \equiv \frac{1}{2}\sum_{r=1}^{2}\sum_{s=1}^{2}|\mathcal{M}_{rs}|^2\,. \tag{12}$$

Using the fact that γ^0 is real & symmetric and that $(\gamma^0)^2 = I$, we can compute

$$\begin{aligned}
(\mathcal{M}_{rs})^* &= [\bar{u}_s(p')\,\Gamma\,u_r(p)]^* \\
&= \left[u_s^\dagger(p')\gamma^0\,\Gamma\,u_r(p)\right]^* \\
&= u_s^{\mathrm{T}}(p')\gamma^0\,\Gamma^*\,u_r^*(p) \\
&= u_r^\dagger(p)\,\Gamma^\dagger\,\gamma^0 u_s(p') \\
&= u_r^\dagger(p)\,\left(\gamma^0\right)^2\,\Gamma^\dagger\,\gamma^0 u_s(p') \\
&= \bar{u}_r(p)\,\left(\gamma^0\Gamma^\dagger\gamma^0\right)u_s(p') \\
&= \bar{u}_r(p)\,\tilde{\Gamma}\,u_s(p')\,,
\end{aligned} \tag{13}$$

where we have defined

$$\tilde{\Gamma} \equiv \gamma^0\Gamma^\dagger\gamma^0\,.$$

Using eqs. 11 and 13 and explicitly writing out the spinor indices as $\alpha, \beta, \gamma, \delta$, we obtain

$$\begin{aligned}
X &= \frac{1}{2}\sum_{r=1}^{2}\sum_{s=1}^{2}[\bar{u}_s^\alpha(p')\Gamma_{\alpha\beta}u_s^\beta(p)]\,[\bar{u}_r^\gamma(p)\tilde{\Gamma}_{\gamma\delta}u_s^\delta(p')] \\
&= \frac{1}{2}\left(\sum_{s=1}^{2}u_s^\delta(p')\bar{u}_s^\alpha(p')\right)\Gamma_{\alpha\beta}\left(\sum_{r=1}^{2}u_s^\beta(p)\bar{u}_s^\gamma(p)\right)\tilde{\Gamma}_{\gamma\delta} \\
&= \frac{1}{2}\,\mathrm{Tr}\left[\frac{\not{p}'+m}{2m}\Gamma\frac{\not{p}+m}{2m}\tilde{\Gamma}\right]\,,
\end{aligned} \tag{14}$$

where we have used the completeness relation

$$\sum_{r=1}^{2}u_s(p)\bar{u}_s(p) = \frac{\not{p}+m}{2m}\,.$$

Similar calculations can be applied to processes involving anti-leptons using the identity

$$\sum_{r=1}^{2} v_s(p)\bar{v}_s(p) = \frac{\not{p} - m}{2m} \,.$$

Similarly, we should consider the polarizations of external photons as well as leptons. For instance, consider a process with one final external photon. Then, the unpolarized cross-section is proportional to the quantity

$$X \equiv \sum_r |\mathcal{M}_r|^2 = \sum_r |\epsilon_r^\mu(k)\mathcal{M}_\mu|^2 \,, \tag{15}$$

where $\epsilon_r^\mu(k)$ is the polarization 4-vector corresponding to a polarization state r and a wave 4-vector k. Note that $k^0 = |\boldsymbol{k}|$.

Since the quantities $\mathcal{M}$ and therefore X are gauge-independent, it is convenient to work with the *Lorenz gauge*. That is, we let the 4-potential A^μ of the external photon satisfy the condition

$$\partial_\mu A^\mu = 0 \,, \tag{16}$$

which describes a simple transverse wave, in agreement with the common intuition of free real radiation.

Because the quantity $\mathcal{M}_r = \epsilon_r^\mu \mathcal{M}_\mu$ is gauge-invariant and we chose a transverse-wave gauge, we have the property

$$k^\mu \mathcal{M}_\mu = 0 \,, \tag{17}$$

and we can choose a coordinate system such that $k^\mu = (\kappa, 0, 0, \kappa)$.

Then, from eq. 15, we can conclude that the unpolarized cross-section must be proportional to

$$\begin{aligned} \sum_{r=1}^{2} |\mathcal{M}_r|^2 &= \mathcal{M}_\mu \mathcal{M}_\nu^* \sum_{r=1}^{2} \epsilon_r^\mu \epsilon_r^\nu \\ &= -\mathcal{M}^\mu \mathcal{M}_\mu^* \,, \end{aligned} \tag{18}$$

where we have used the gauge condition eq. 17 and the completeness relation for real photons, i.e.,

$$\sum_{r=1}^{2} \epsilon_r^\mu \epsilon_r^\nu = -g^{\mu\nu} + \frac{1}{2}\left(k^\mu \tilde{k}^\nu + \tilde{k}^\mu k^\nu\right) \,,$$

where $\tilde{k}^\mu = (\kappa^{-1}, 0, 0, -\kappa^{-1})$.

Now, we can use these results to calculate some basic QED cross-sections.

3 $e^+e^- \to l^+l^-$ Process ($l \neq e$)

Consider the collision process in which an electron-positron collides and annihilates to produce a lepton-anti-lepton pair. If the polarizations of the external leptons are all specified, this can be written as

$$e^+(\boldsymbol{p}_1, r_1) + e^-(\boldsymbol{p}_2, r_2) \longrightarrow l^+(\boldsymbol{p}_1', s_1) + l^-(\boldsymbol{p}_2', s_2) \tag{19}$$

where the $\boldsymbol{p}$'s denote 3-momenta, r's and s's denote spin states, and the primes label the final state.

For simplicity, we will let the final state leptons (a lepton and an anti-lepton) be muons or tauons, but not electrons.

Then, the first-order Feynman diagram corresponding to the process 19 is given by fig. 1. Its invariant amplitude is

$$\mathcal{M}(r_1, r_2, s_1, s_2) = ie^2 \underbrace{\left[\bar{u}_{l^-}(p_2', s_2)\gamma_\mu v_{l^+}(p_1', s_1)\right]}_{\text{final lepton vertex}} \underbrace{\frac{1}{(p_1+p_2)^2}}_{\text{photon propagator}} \underbrace{\left[\bar{v}_{e^+}(p_1, r_1)\gamma^\mu u_{e^-}(p_2, r_2)\right]}_{\text{initial electron vertex}}, \tag{20}$$

and the complex conjugate is

$$\mathcal{M}^*(r_1, r_2, s_1, s_2) = -ie^2 \left[\bar{v}_{l^+}(p_1', s_1)\gamma_\mu u_{l^-}(p_2', s_2)\right] \frac{1}{(p_1+p_2)^2} \left[\bar{u}_{e^-}(p_2, r_2)\gamma^\mu v_{e^+}(p_1, r_1)\right] ,$$

where we have used the γ-matrix identity: $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$.

Averaging over r's and summing over s's (generalizing eq. 14), we obtain the quantity proportional to the unpolarized cross-section

$$\begin{aligned} X &\equiv \frac{1}{4}\sum_{r_1}\sum_{r_2}\sum_{s_1}\sum_{s_2} |\mathcal{M}(r_1, r_2, s_1, s_2)|^2 \\ &= \frac{e^4}{4(p_1+p_2)^4} A^l_{\mu\nu} B_e^{\mu\nu} \\ &= \frac{e^4}{4(p_1+p_2)^4} \mathrm{Tr}\left[\frac{\not{p}_2' + m_l}{2m_l}\gamma_\mu \frac{\not{p}_1' - m_l}{2m_l}\gamma_\nu\right] \mathrm{Tr}\left[\frac{\not{p}_1 - m_e}{2m_e}\gamma^\mu \frac{\not{p}_2 + m_e}{2m_e}\gamma^\nu\right] . \end{aligned} \tag{21}$$

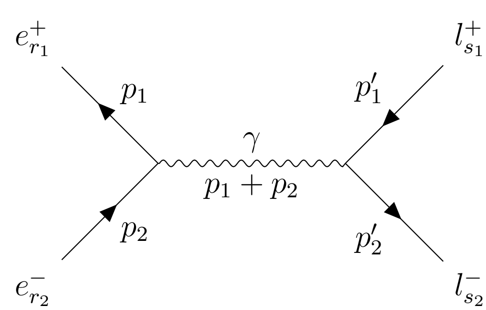

Figure 1: Lowest-order Feynman diagram for the process $e^+e^- \to l^+l^-$ with complete polarization-specification.

Now, we use γ-matrix trace theorems to calculate the quantities in eq. 21. In particular,

$$\begin{gathered}
\mathrm{Tr}(\gamma^{\mu_1}...\gamma^{\mu_n}) = 0 \qquad \text{if } n = \text{odd}\,, \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}\,, \\
\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)\,.
\end{gathered}$$

Then, we get

$$\begin{aligned}
A^l_{\mu\nu} &= \mathrm{Tr}\left[\frac{\not{p}'_2 + m_l}{2m_l}\gamma_\mu\frac{\not{p}'_1 - m_l}{2m_l}\gamma_\nu\right] \\
&= \frac{1}{4m_l^2}\,\mathrm{Tr}\left[\not{p}'_2\gamma_\mu\not{p}'_1\gamma_\nu - m_l^2\gamma_\mu\gamma_\nu\right] \\
&= \frac{1}{4m_l^2}\left[p_2'^\rho p_1'^\sigma\,\mathrm{Tr}\left(\gamma_\rho\gamma_\mu\gamma_\sigma\gamma_\nu\right) - m_l^2\,\mathrm{Tr}\left(\gamma_\mu\gamma_\nu\right)\right] \\
&= \frac{1}{4m_l^2}\left[4p_2'^\rho p_1'^\sigma\left(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\sigma}g_{\mu\nu} + g_{\rho\nu}g_{\mu\sigma}\right) - 4m_l^2 g_{\mu\nu}\right] \\
&= \frac{1}{m_l^2}\left[p'_{1\mu}p'_{2\nu} + p'_{2\mu}p'_{1\nu} - (m_l^2 + p'_1p'_2)g_{\mu\nu}\right]
\end{aligned}$$

and similarly‘

$$B_e^{\mu\nu} = \frac{1}{m_e^2}\left[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - (m_e^2 + p_1p_2)g^{\mu\nu}\right]\,.$$

Substituting into eq. 21, we get

$$X = \frac{e^4}{2m_e^2m_l^2(p_1+p_2)^4}\left[(p_1p'_1)(p_2p'_2) + (p_1p'_2)(p_2p'_1) + m_e^2p'_1p'_2 + m_l^2p_1p_2 + 2m_e^2m_l^2\right]$$

Now, to simplify this expression further, we consider a specific frame of reference: the center-of-momentum frame. In this frame, $\boldsymbol{p}_1 = -\boldsymbol{p}_2$, $\boldsymbol{p}'_1 = -\boldsymbol{p}'_2$, and $E_1 = E_2 = E'_1 = E'_2 = E$. The kinematics of $e^+e^- \to l^+l^-$ process is simply described by the angle θ between $\boldsymbol{p}_1$ & $\boldsymbol{p}'_1$ and the magnitudes of the 3-momenta $p \equiv |\boldsymbol{p}|$ and $p' \equiv |\boldsymbol{p}'|$. In summary, we have the following results:

$$\begin{gathered}
p_1p'_1 = p_2p'_2 = E^2 - pp'\cos\theta\,, \qquad p_1p'_2 = p_2p'_1 = E^2 + pp'\cos\theta\,, \\
p_1p_2 = E^2 + p^2\,, \qquad p'_1p'_2 = E^2 + p'^2\,, \\
(p_1+p_2)^2 = 4E^2\,.
\end{gathered} \tag{22}$$

Therefore, we obtain the final unpolarized differential cross-section for the CoM frame:

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\boldsymbol{\Omega}}\right)_{\text{CoM}} &= \frac{1}{64\pi^2(E_1+E_2)^2}\frac{|\boldsymbol{p}_1'|}{|\boldsymbol{p}_1|}\left(\prod_l 2m_l\right) X \\
&= \frac{1}{256\pi^2E^2}\frac{p'}{p}16m_e^2m_l^2\frac{e^4}{32m_e^2m_l^2E^4} \\
&\quad\times\left[2E^4+2p^2p'^2\cos^2\theta+m_e^2(E^2+p'^2)+m_l^2(E^2+p^2)+2m_e^2m_l^2\right] \\
&\approx \frac{\alpha^2}{16E^4}\left(\frac{p'}{E}\right)\left[E^2+m_l^2+p'^2\cos^2\theta\right] ,
\end{aligned}
\tag{23}
$$

where $\alpha \equiv e^2/(4\pi)$ and we have made the approximation $m_e \ll m_l \leq E$ and therefore $p^2 \approx E^2$.

The total cross-section is obtained by integrating over the whole 4π steradian solid angle:

$$
\sigma_{\text{CoM}} = \frac{\pi\alpha^2}{4E^4}\left(\frac{p'}{E}\right)\left[E^2+m_l^2+\frac{1}{3}p'^2\right] . \tag{24}
$$

In even more high-energy cases, we have $E \gg m_l$ which also implies $p' \approx E$. Therefore,

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\boldsymbol{\Omega}}\right)_{\text{CoM}} &= \frac{\alpha^2}{16E^2}\left(1+\cos^2\theta\right) , \\
\sigma_{\text{CoM}} &= \frac{\pi\alpha^2}{3E^2} .
\end{aligned}
$$

4 Bhabha Scattering: $e^+e^- \to e^+e^-$ Process

In the previous section, we ignored the case where the final state lepton pair is an electron-positron pair. This is because there are *two* lowest-order Feynman diagrams that contribute to this process. In addition to the pair annihilation-creation process discussed previously, there is also elastic scattering of the electron and the positron by simply exchanging a photon. See fig. 2.

The invariant amplitude for process (a) was already discussed in the previous section:

$$
\mathcal{M}_{\text{a}} = ie^2\left[\bar{u}(p_2', s_2)\gamma_\mu v(p_1', s_1)\right]\frac{1}{(p_1+p_2)^2}\left[\bar{v}(p_1, r_1)\gamma^\mu u(p_2, r_2)\right] . \tag{25}
$$

The invariant amplitude for process (b) is:

$$
\mathcal{M}_{\text{b}} = -ie^2\underbrace{\left[\bar{u}(p_2', s_2)\gamma_\mu u(p_2, r_2)\right]}_{\text{lower vertex}}\underbrace{\frac{1}{(p_1'-p_1)^2}}_{\text{photon propagator}}\underbrace{\left[\bar{v}(p_1, r_1)\gamma^\mu v(p_1', s_1)\right]}_{\text{upper vertex}} . \tag{26}
$$

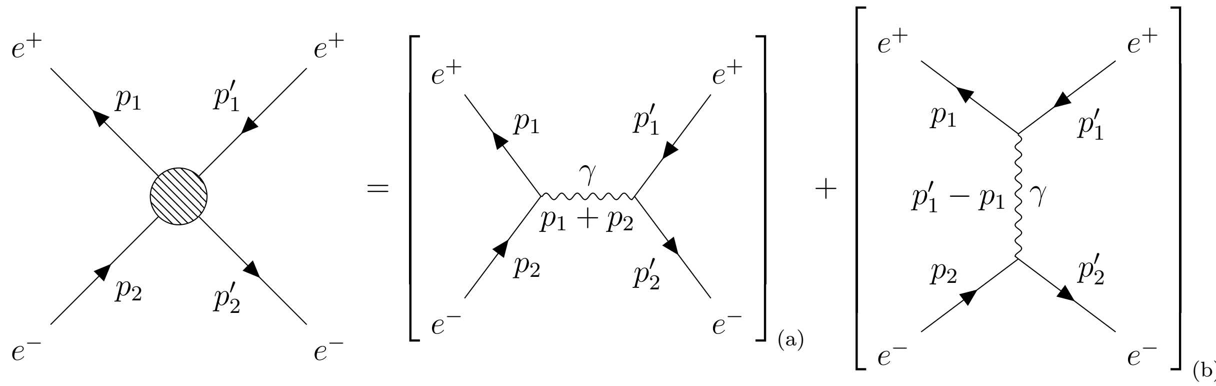

Figure 2: Two lowest-order Feynman diagrams for the $e^+e^- \to e^+e^-$ process.
(a) Annihilation-creation process, discussed previously.
(b) Elastic scattering by photon exchange.

Hence, the unpolarized differential cross-section is now proportional to the quantity

$$\begin{aligned} X &\equiv \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{\text{a}} + \mathcal{M}_{\text{b}}|^2 \\ &= \frac{1}{4}\sum_{\text{spins}} \left(|\mathcal{M}_{\text{a}}|^2 + |\mathcal{M}_{\text{b}}|^2 + \mathcal{M}_{\text{a}}\mathcal{M}_{\text{b}}^* + \mathcal{M}_{\text{a}}^*\mathcal{M}_{\text{b}}\right) . \end{aligned} \tag{27}$$

Again, we choose to work in the CoM frame so that we have all the identities in eq. 22 and additionally

$$|\boldsymbol{p}_1| = |\boldsymbol{p}_2| = p = |\boldsymbol{p}_1'| = |\boldsymbol{p}_2'| = p' ,$$

and consider the relativistic case so that $E \gg m_e$ (i.e. $m_e/E \approx 0$) and $p = p' \approx E$.

Then, we get

$$\begin{aligned} X_{\text{aa}} &\equiv \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{\text{a}}|^2 \\ &\approx \frac{e^4}{16 m_e^4}\left[1 + \cos^2\theta\right] , \end{aligned} \tag{28}$$

which follows from the result that we saw in the previous section.

Similar process shows that

$$\begin{aligned} X_{\text{bb}} &\equiv \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{\text{b}}|^2 \\ &\approx \frac{e^4}{2 m_e^4 (p_1' - p_1)^4}\left[(p_1p_1')(p_2p_2') + (p_1p_2')(p_2p_1')\right] \\ &\approx \frac{e^4}{8 m_e^4 \sin^4(\theta/2)}\left[1 + \cos^4\frac{\theta}{2}\right] . \end{aligned} \tag{29}$$

The last term, which is the cross-term between processes (a) and (b), is more complicated. With higher-power trace theorems, we obtain

$$
\begin{aligned}
X_{\mathrm{ab}} &\equiv \frac{1}{4}\sum_{\mathrm{spins}} \mathcal{M}_{\mathrm{a}}\mathcal{M}_{\mathrm{b}}^{*} \\
&= \frac{-e^4}{4(p_1+p_2)^2(p_1'-p_1)^2}\sum_{\mathrm{spins}} \left[\bar{u}(p_2',s_2)\gamma_\mu v(p_1',s_1)\right]\left[\bar{v}(p_1,r_1)\gamma^\mu u(p_2,r_2)\right] \\
&\qquad\qquad \times \left[\bar{v}(p_1',s_1)\gamma_\nu v(p_1,r_1)\right]\left[\bar{u}(p_2,r_2)\gamma^\nu u(p_2',s_2)\right] \\
&= \frac{-e^4}{4(p_1+p_2)^2(p_1'-p_1)^2}\sum_{\mathrm{spins}} \left[\bar{u}(p_2',s_2)\gamma_\mu v(p_1',s_1)\right]\left[\bar{v}(p_1',s_1)\gamma_\nu v(p_1,r_1)\right] \\
&\qquad\qquad \times \left[\bar{v}(p_1,r_1)\gamma^\mu u(p_2,r_2)\right]\left[\bar{u}(p_2,r_2)\gamma^\nu u(p_2',s_2)\right] \\
&= \frac{-e^4}{4(p_1+p_2)^2(p_1'-p_1)^2}\,\mathrm{Tr}\left[\frac{\not{p}_2'+m_e}{2m_e}\gamma_\mu\frac{\not{p}_1'-m_e}{2m_e}\gamma_\nu\frac{\not{p}_1-m_e}{2m_e}\gamma^\mu\frac{\not{p}_2+m_e}{2m_e}\gamma^\nu\right] \\
&\approx \frac{-e^4}{64m_e^4(p_1+p_2)^2(p_1'-p_1)^2}\,\mathrm{Tr}\left[\not{p}_2'\gamma_\mu\not{p}_1'\gamma_\nu\not{p}_1\gamma^\mu\not{p}_2\gamma^\nu\right] \\
&= \frac{-e^4}{8m_e^4\sin^2(\theta/2)}\cos^4\frac{\theta}{2}\,.
\end{aligned} \tag{30}
$$

Therefore, the unpolarized differential cross-section in the highly-relativistic CoM frame is

$$
\begin{aligned}
\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\boldsymbol{\Omega}}\right)_{\mathrm{CoM}} &= \frac{1}{64\pi^2(E_1+E_2)^2}\frac{|\boldsymbol{p}_1'|}{|\boldsymbol{p}_1|}(2m_e)^4\left[X_{\mathrm{aa}}+X_{\mathrm{bb}}+X_{\mathrm{ab}}+X_{\mathrm{ab}}^{*}\right] \\
&= \frac{\alpha^2}{8E^2}\left[\frac{1+\cos^2\theta}{2}+\frac{1+\cos^4(\theta/2)}{\sin^4(\theta/2)}-2\frac{\cos^4(\theta/2)}{\sin^2(\theta/2)}\right]\,.
\end{aligned} \tag{31}
$$

Note that this quantity diverges to positive infinity as $\theta \to 0$ due to the X_{bb} contribution. In other words, the differential-cross section is infinite in the co-linear forward direction. This is because the 4-momentum of the photon exchanged between the electron and the positron $k^\mu \equiv (p_1' - p_1)^\mu$ goes to zero and the photon propagator in eq. 26 diverges.

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