Important Invariants

In Γ space, suppose that ensemble points at time t fill phase space $d^n q_s d^n p_s$ and at time t' fill phase space $d^n q_s' d^n p_s'$

The conservation of ensemble points in phase space implies

$$\rho_{ens} d^n q_s' d^n p_s' = \rho_{ens}' d^n q_s d^n p_s \qquad (13b)$$

and Liouvilles theorem $\Rightarrow$ $\rho_{ens}(q_s, p_s, t) = \rho_{ens}'(q_s', p_s', t')$

$$\Rightarrow \quad d^n q_s' d^n p_s' = d^n q_s d^n p_s$$ (assuming elastic collisions)

For μ space, Liouvilles theorem holds only for collisionless systems. In this case

$$f\, d^3x\, d^3p = f'\, d^3x'\, d^3p'$$ (number conservation of particles)

Liouvilles thm $\Rightarrow$ $f = f'$

$$\Rightarrow \quad d^3x\, d^3p = d^3x'\, d^3p'$$

(volumes in which fixed # particles are contained)

We will use these later particularly

(14a)

Collisional Boltzmann Equation

Need to modify purely collisionless Boltzmann equation to include interactions between particles

We consider the case of a dilute gas

$n a^3 << 1$ (small particle radius $= a$ compared to interparticle spacing).

and no long-range interactions between particles.

Now the collisionless Boltzmann equation says that $f(\vec{x},\vec{p},t)$ does not change along the trajectory of a particle. Collisions can change this by bumping particles to different velocities, thus increasing or decreasing the number of particles in a given μ-space

thus.

$$\frac{Df}{Dt} d^3x\, d^3u = C_{in} - C_{out}$$

$C_{in}, C_{out} \equiv$ rates at which particles enter or leave $d^3x\, d^3u$ from collisions

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$2u_1'u_0' = 2u_1u$

We consider elastic collisions:

$$\vec{u} + \vec{u}_1 = \vec{u}' + \vec{u}_1' \qquad \text{momentum cons}$$

( )

$\vec{u}, \vec{u}_1$ = particle velocities before collisions (mom)

$\vec{u}', \vec{u}_1'$ = velocities after collision

$(u_1 - u_2)^2 = (u_1' - u_2')^2$

$u_1^2 + u_2^2 = u_1'^2 + u_2'^2$

$-2u_1u_2 + 2u_1'u_2'$

$$\tfrac{1}{2}|\vec{u}|^2 + \tfrac{1}{2}|\vec{u}_1|^2 = \tfrac{1}{2}|\vec{u}'|^2 + \tfrac{1}{2}|\vec{u}_1'|^2 \qquad \text{energy cons}$$

$(u_1 - u_0)^2 = (u' - u_1')^2$

these equations provide <u>4</u> equations for <u>6</u> unknowns. $(\vec{u}'^{\,3}, \vec{u}_1'^{\,3})$ final velocities

The remaining constraints come from:

1) <u>coplanarity of $\vec{u}', \vec{u}_1', \vec{u}, \vec{u}_1$</u> for radial force of interactions (e.g. coulomb collisions) - eliminates

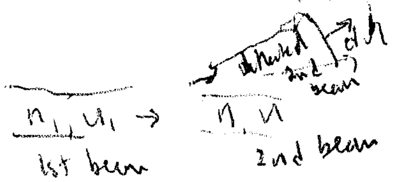
2) <u>impact parameter</u>, which gives the $\measuredangle$ of deflection. This comes from microphysics of interaction

Statistically, # 2) is modeled by differential cross section. We assume its given and show how dynamics of system can then be studied:

Consider beam of particles with number density n_1 and velocity $\vec{u}_1$, colliding with beam having number density n and velocity $\vec{u}$. The latter beam sees particle flux $I = |\vec{u} - \vec{u}_1| n_1$ from first beam

(number per area per time)

Define $\delta n_c \equiv \frac{\#\text{ collisions}}{\text{time}\cdot\text{volume}}$ that deflect particles from second beam into solid angle $d\Omega$, by interaction with first beam:



$$\delta n_c = \underbrace{(n)}_{\substack{n \text{ of second}\\ \text{beam}}}\underbrace{(I)}_{\substack{\text{flux of } n_1 \text{ that}\\ n\text{-beam is exposed to}}}\underbrace{d\Omega}_{\text{solid } \measuredangle}\underbrace{\sigma(\vec{u}, u_1 | u', u_1')}_{\substack{\text{differential scattering}\\ \text{cross section}}}$$

$\delta n_c = \frac{\#}{\text{vol, time, sol. } \measuredangle}$

(individual interactions are reversible for elastic scattering so that

$$\sigma(\vec{u}, u_1 | u_1', u') = \sigma(\vec{u}_1', \vec{u}' | \vec{u}, \vec{u}_1)$$
)

Now

since $n = f(\vec{x}, \vec{u}, t)\, d^3\vec{u}$ = number per volume

and $n_1 = f(\vec{x}, u_1, t)\, d^3 u_1$

and $I = |\vec{u} - \vec{u}_1| n_1 = |u_1 - u_1| f(\vec{x}, \vec{u}_1, t)\, d^3 u_1$

$$\delta n_c = \sigma(u, u_1 | u', u_1')\, |\vec{u} - \vec{u}_1|\, f(x, u, t)\, f(x, u_1, t)\, d\Omega\, d^3u\, d^3u_1$$

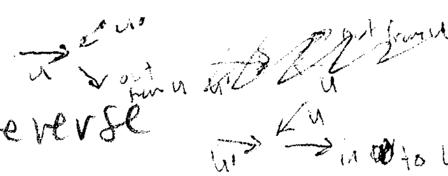
since $C_{out} = \frac{\#\text{ collisions}}{\text{time}}$ in phase volume $d^3x\, d^3u$,

$$C_{out} = d^3x \underbrace{\int \delta n_c (d\Omega\, d^3u_1)}_{} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(u, u_1 | u', u_1')\, |\vec{u} - \vec{u}_1|\, f(x, \vec{u}, t)\, f(x, \vec{u}_1, t)$$

means $d^3x \int_{d\Omega,\, d^3u_1} \delta n_c$

i.e. multiply δn_c by d^3x and integrate over d^3u_1, $d\Omega$

$\rightarrow$



To get C_{in} consider reverse collisions; that is replace $u' \leftrightarrow u$ and $u_1' \leftrightarrow u_1$ straight away we have:

$$C_{in} = d^3x\, d^3u' \int d^3u_1' \int d\Omega\, \sigma(u, u_1 | u', u_1')\, |u'-u_1'|\, f(x, u', t) f(x, u_1', t)$$

But: ① conservation of momentum & energy for collisions $\Rightarrow$ $|u - u_1| = |u' - u_1'|$

and ② Earlier we proved (eqn 13b) that phase space measures at any time are equal (from Liouville's Thm + conservation of particle number & elastic collision assumption) thus for 2-particle phase space

$d^3u\, d^3u_1 = d^3u'\, d^3u_1'$. ③ we also argued $\sigma(u, u_1 | u', u_1') = \sigma(u', u_1' | u, u_1)$. Thus ①, ②, ③

$$\Rightarrow C_{in} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(u', u_1' | u, u_1)\, |u - u_1|\, f(x, u', t) f(x, u_1', t)$$

comparing to C_{out} we then combine to get:

$$\frac{Df}{Dt} d^3x\, d^3u = C_{in} - C_{out} = d^3x\, d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega) (f' f_1' - f f_1)$$

(where $f' \equiv f(u')$ and $f_1' \equiv f(u_1')$
$f_1 \equiv f(u_1)$; $f' = f(u')$) $\longrightarrow$

we thus have

$$\boxed{\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \vec{u}\cdot\nabla f + \frac{\vec{F}}{m}\cdot\nabla_{u} f = \underbrace{\int d^3u_1 \int d\Omega\, |u-u_1|\,\sigma(\Omega)\,(f'f_1' - ff_1)}_{\text{Collisional Boltzmann eqn}}} \quad (14)$$

$\vec{F} = m\dot{\vec{u}}$ is any force field that particles experience e.g. <u>gravity</u>

to recap: right side measures effects of Collisions on distribution function for a <u>dilute</u> gas. (dilute because we assumed only binary collisions)

<u>Maxwellian Distribution</u>

Uniform classical gas relaxes to maxwell dist. This can be derived from above collisional Boltz. eqn: Consider case when $\vec{F}$ term is negligible, and f is independent of time and space (ie. in equilibrium).

Boltz eqn $\Rightarrow$

$$f f_1 = f' f_1'$$

or
$$\log f(u) + \log f(u_1) = \log f'(u) + \log f_1'(u_1)$$

$\rightarrow$

If $\chi(u)$ is a conserved quantity then

$$\underbrace{\chi(u) + \chi(u_1)}_{\text{before}} = \underbrace{\chi(u') + \chi(u_1')}_{\text{after collision}}$$

Since this has same form of previous equation we must be able to write $\log f(u)$ as a linear combination of $\chi(u)$

That is:

sum over all conserved quantities

$$\log f(u) = C_0 + \sum_s C_r \chi_s(\vec{u}) \qquad (C_0, C_r \text{ are constants})$$

As energy & the 3 momenta are the (complete) relevant quantities here:

$$\log f(\vec{u}) = C_0 + C_1 \vec{u}^2 + C_{2x} u_x + C_{2y} u_y + C_{2z} u_z$$

$$\Rightarrow \log f(\vec{u}) = -B(\vec{u} - \vec{u}_0)^2 + \log A$$

where $C_0, C_1, C_{2x}, C_{2y}, C_{2z}$ have been replaced by

exponentiate $B, A, u_{0x}, u_{0y}, u_{0z}$,

$$\Rightarrow f(u) = A e^{-B(\vec{u} - u_0)^2}$$

$$n = \int d^3u\, f(u) \Rightarrow A = \left(\frac{B}{\pi}\right)^{3/2} n$$

$\rightarrow$

(22)

$\Rightarrow$

$$f(u) = \left(\frac{B}{\pi}\right)^{3/2} n\, e^{-B(\vec{u}-u_0)^2}$$

note that

$$\langle \vec{u} \rangle = \frac{1}{n}\int f(u)\, \vec{u}\, d^3u = \left(\frac{B}{\pi}\right)^{3/2} \int d^3\vec{u}\,(u+U_0)\, e^{-Bu^2}$$

(where $\vec{u} \to \vec{u} + \vec{U}_0$ change of variables was used)

$$= \vec{U}_0 \left(\frac{B}{\pi}\right)^{3/2} \int d^3u\, e^{-Bu^2} = \vec{U}_0$$

$\Rightarrow$ non-zero $\vec{U}_0$ implies a mean streaming motion.

$I_n = \int_{-\infty}^{\infty} e^{-Bu^2} u^n\, du$

$I_{n+2} = -\frac{\partial I_n}{\partial B}$

$I_0 = \sqrt{\frac{\pi}{B}}$

$I_2 = \frac{1}{2}\sqrt{\frac{\pi}{B^3}}$

if we go to frame in which $\vec{U}_0 = 0$ and consider system of temperature T, then $B = \frac{m}{2k_BT}$

and $$f(u) = n\left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{m\vec{u}^2}{2k_BT}\right]$$

Maxwell Boltzmann

is a soln to steady-state Boltzmann equation

NOT SURPRISING!

Conservation equations

$$\chi + \chi_1 = \chi' + \chi_1' \qquad (*)$$

for conserved quantity $\chi(\vec{x}, \vec{u})$ before & after collisions

now let us derive the equation for the averaged χ

this is important for eventually deriving the hydrodynamic fluid eqns:

we multiply the collisional Boltzman equation by χ. The result for the RHS after integrating is

$$= \int d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)\, |\vec{u}-u_1| (f'f_1' - ff_1)\, \chi(\vec{u}, \vec{x})$$

$$= \frac{1}{2} \int d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)\, |\vec{u}-u_1| (f'f_1' - ff_1) (\chi(\vec{u}, x) + \chi(\vec{u}_1, x))$$

since the RHS of collision Botz. Eqn is symmetric in $\vec{u}_1 \leftrightarrow \vec{u}$.

We can also go further:

$$= \frac{1}{4} \int d^3u \int d^3u_1 \int d\Omega\, \sigma(\Omega)\, |\vec{u}-u_1| (f'f_1' - ff_1) (\chi(\vec{u}, x) + \chi(\vec{u}_1, x) - \chi(\vec{u}', x) - \chi(u_1', x))$$

because the collision integral in Boltzmann eqn is antisymmetric in $u' \leftrightarrow u$, $u_1' \to u'$. But from $(*)$ this RHS now $= 0$!

$\rightarrow$

the left side of (14) when multiplied by $\underline{\chi}$ & integrated is then $=0$ $\Rightarrow$ we have

$$\int d^3u\, \underline{\chi} \left(\frac{\partial f}{\partial t} + u_i \frac{\partial f}{\partial x_i} + \frac{F_i}{m} \frac{\partial f}{\partial u_i} \right) = 0$$

manipulation using chain rule gives

$$\partial_t \int d^3u\, \underline{\chi} f + \frac{\partial}{\partial x_i} \int d^3u\, \underline{\chi} u_i f - \int d^3u\, u_i f \frac{\partial \underline{\chi}}{\partial x_i}$$
$$+ \frac{1}{m} \int d^3u\, \underbrace{\frac{\partial}{\partial u_i} (\underline{\chi} F_i f)}_{0} - \frac{1}{m} \int d^3u\, \frac{\partial \underline{\chi}}{\partial u_i} F_i f - \frac{1}{m} \int d^3u\, \underline{\chi} \frac{\partial F_i}{\partial u_i} f = 0 \qquad (14a)$$

(surface term by Gauss thm)

using the notation $\langle \underline{\chi} \rangle = \frac{1}{n} \int f \underline{\chi}\, d^3u$ with $n = \int f d^3u$, we can write (14a) as

$$\partial_t (n \langle \underline{\chi} \rangle) + \frac{\partial}{\partial x_i} (n \langle u_i \underline{\chi} \rangle) - n \left\langle u_i \frac{\partial \underline{\chi}}{\partial x_i} \right\rangle - \frac{n}{m} \left\langle F_i \frac{\partial \underline{\chi}}{\partial u_i} \right\rangle - \frac{n}{m} \left\langle \frac{\partial F_i}{\partial u_i} \underline{\chi} \right\rangle = 0$$

this tells us how the volume density of any quantity $n \langle \underline{\chi} \rangle$ evolves with time

$\rightarrow$